Polyhedral Clinching Auctions and the AdWords Polytope

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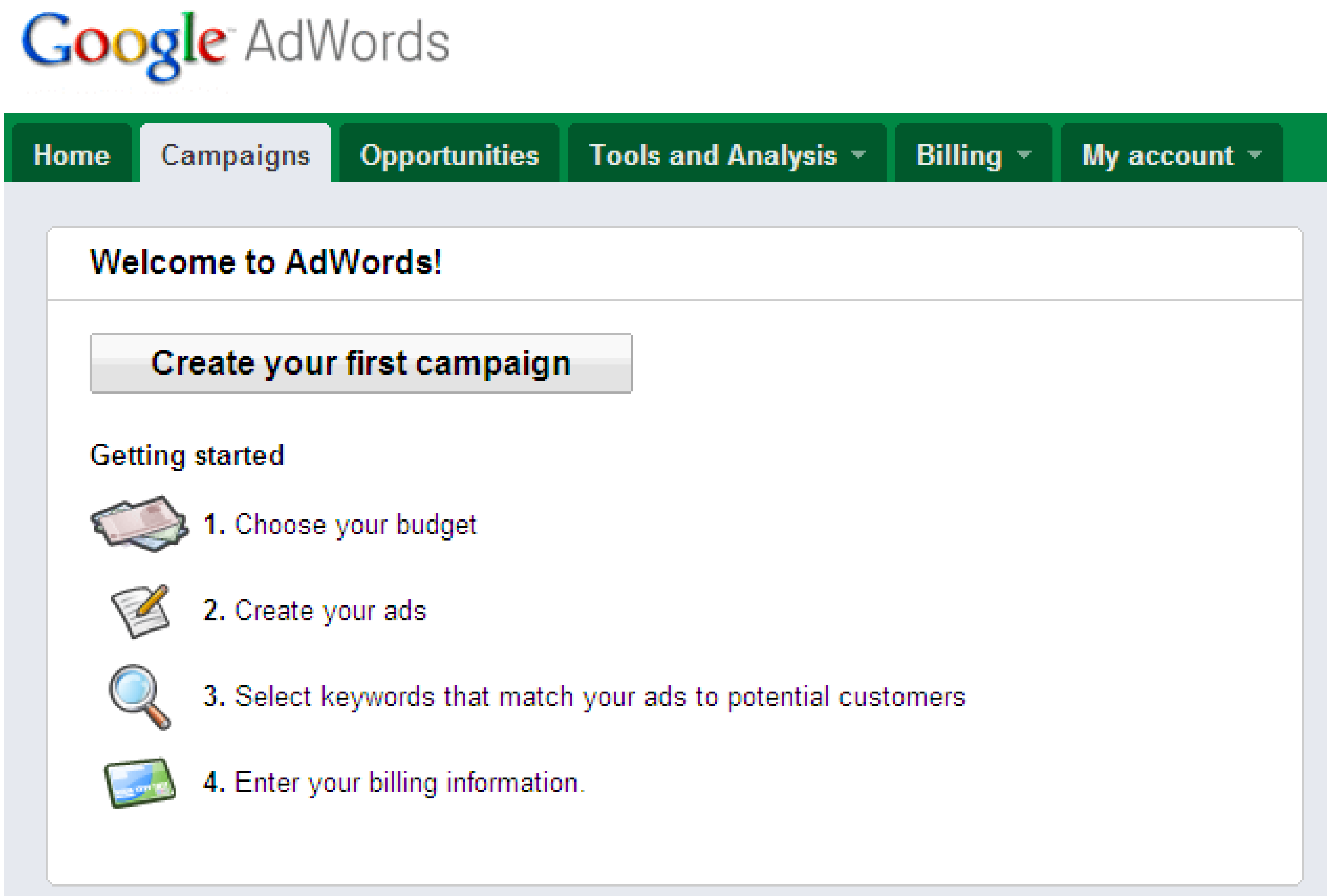
(Cornell University)

Joint work with Gagan Goel and

Vahab Mirrokni (Google NYC)

Creating an Ads campaign …



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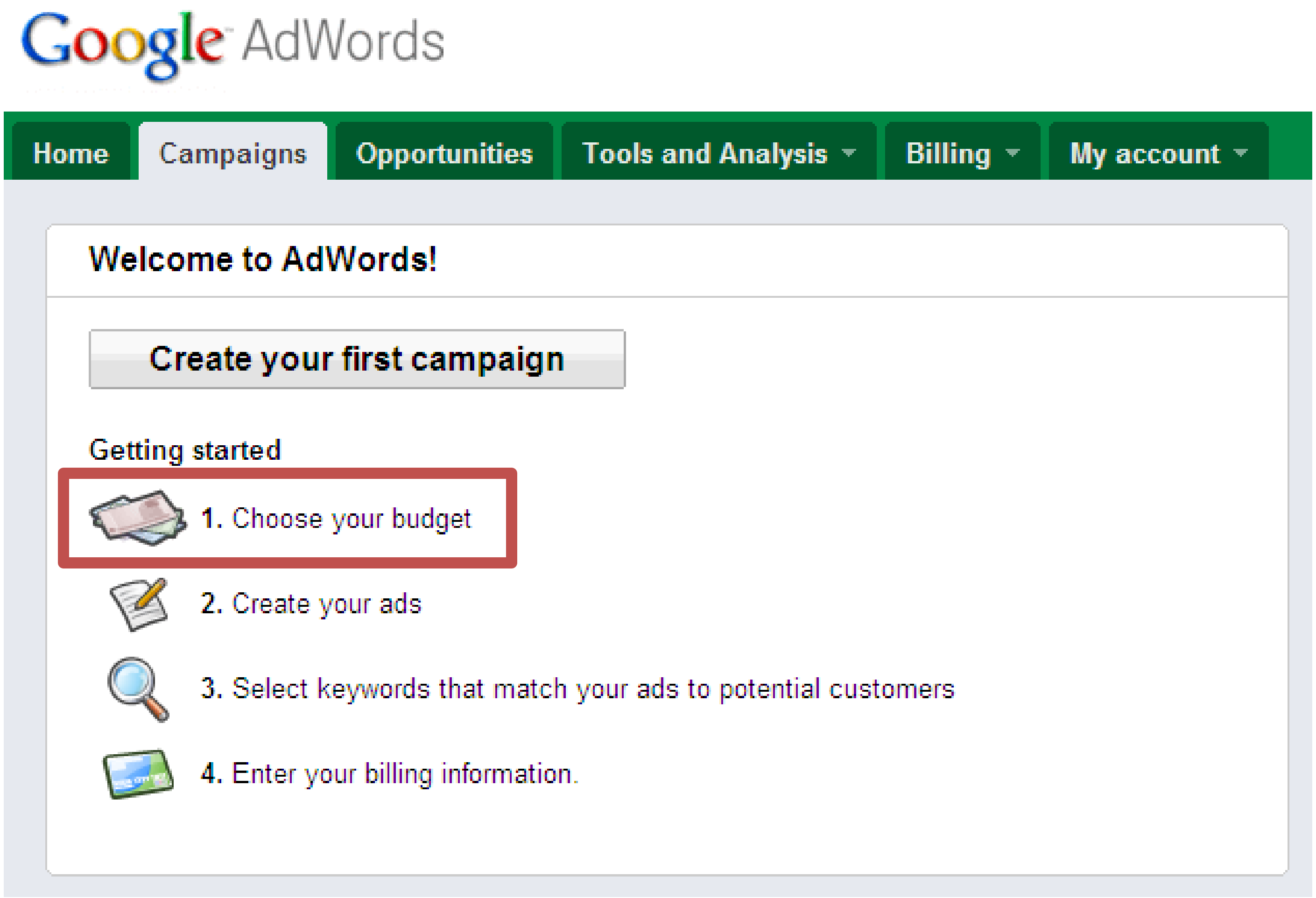

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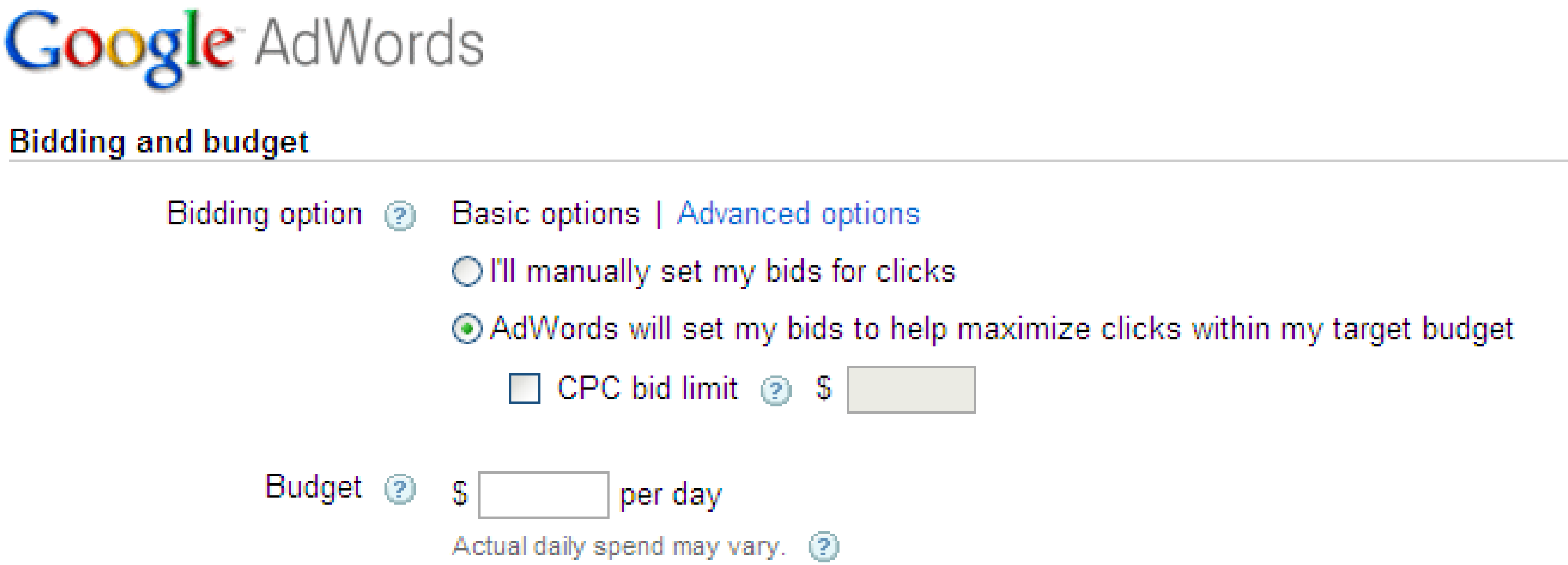

How to deal with budgets in practice ?

VCG, GSP, …

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nice and well studied
auction with good
game-theoretic properties
but without budgets…

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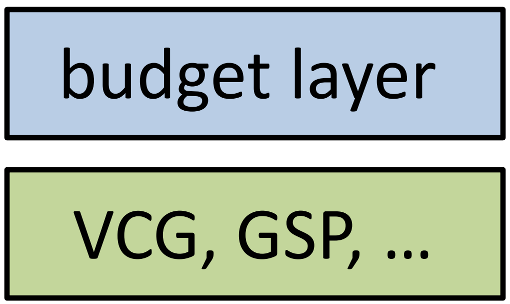

How to deal with budgets in practice ?

engineering fix to adapt the original auction to the budgeted setting.
Original game theoretic analysis is now lost.

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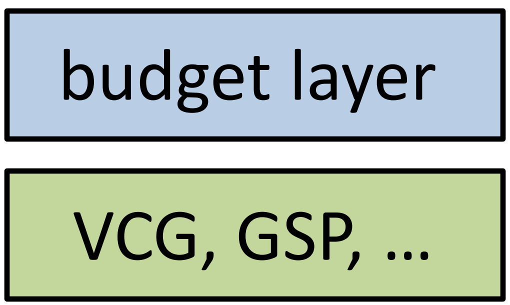

How to deal with budgets in practice ?

Polyhedral Clinching Auction

Goal:

Design an auction for AdWords that supports budgets natively, i.e., budgets are built in the game theoretic analysis

What do we mean by budgets ?

Classical quasi-linear utility function:

$u_i = v_i($  $) - p_i$

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$\quad = -\infty, \qquad\qquad\qquad \text{o.w.}$

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Very well understood: **VCG**, affine maximizers, …

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Surprisingly little is known.

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- Agents report **values** and **budgets** (v_i, B_i)
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(feasible set)

Desirable properties

- Incentive Compatibility:

  $$v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i x_i(v'_i, v_{-i}) - p_i(v'_i, v_{-i})$$

  assumption: budgets B_i are public

- Individual rationality: $v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq 0$

- Pareto optimality:

> An outcome (x,p) is Pareto-optimal if there is no (x',p') such that $u'_i \geq u_i$, $\Sigma p'_i \geq \Sigma p_i$ and at least one of them is strict.

Our main contribution

Solve this problem for a large class of feasible sets **P**.

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Show this is impossible to be extended to general polytopes.

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Show this is impossible to be extended to general polytopes.

Conjecture: scaled polymatroids are the largest class for which this is possible.
(we supply evidence for that)

What do we know about budgets?

[Dobzinski, Lavi, Nisan, FOCS'08]
:: auction for one divisible good

[Fiat, Leonardi, Saia, Sankowski, EC'11]
:: auction for matching markets

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based on the **clinching auctions framework**
[Ausubel, AER'97]

How does it fit in our goal ?

[Dobzinski, Lavi, Nisan, FOCS'08]

P = $\{x \in \mathbb{R}^n_+; \sum_i x_i \leq 1\}$ **Uniform Matroid**

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P = **Transversal Matroid**

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P = **Transversal Matroid**

For AdWords and other more complicated markets, we need to solve it for more generic feasibility constraints **P**

Our Results

We provide an auction with all the desirable properties for any polymatroid **P**.

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- Incentive compatibility
- Individual Rationality
- Budget Feasibility
- Pareto Optimality

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$$\mathbf{P} = \{x \in \mathbb{R}^n_+; \sum_{i \in S} x_i \leq f(S); \forall S \subseteq [n]\}$$

for a submodular function *f*.

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We provide an auction with all the desirable properties for any polymatroid **P**.

Our auction only needs oracle access to the submodular function f.

Our auction has a natural geometric flavor.

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Many applications

Auctions for network design, queuing systems, video on demand, matching markets, internet advertisement, …

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Our results

The set of $(x_1, \ldots, x_n)$ that can be obtained this way form a polymatroid. We call it the **AdWords Polytope**.

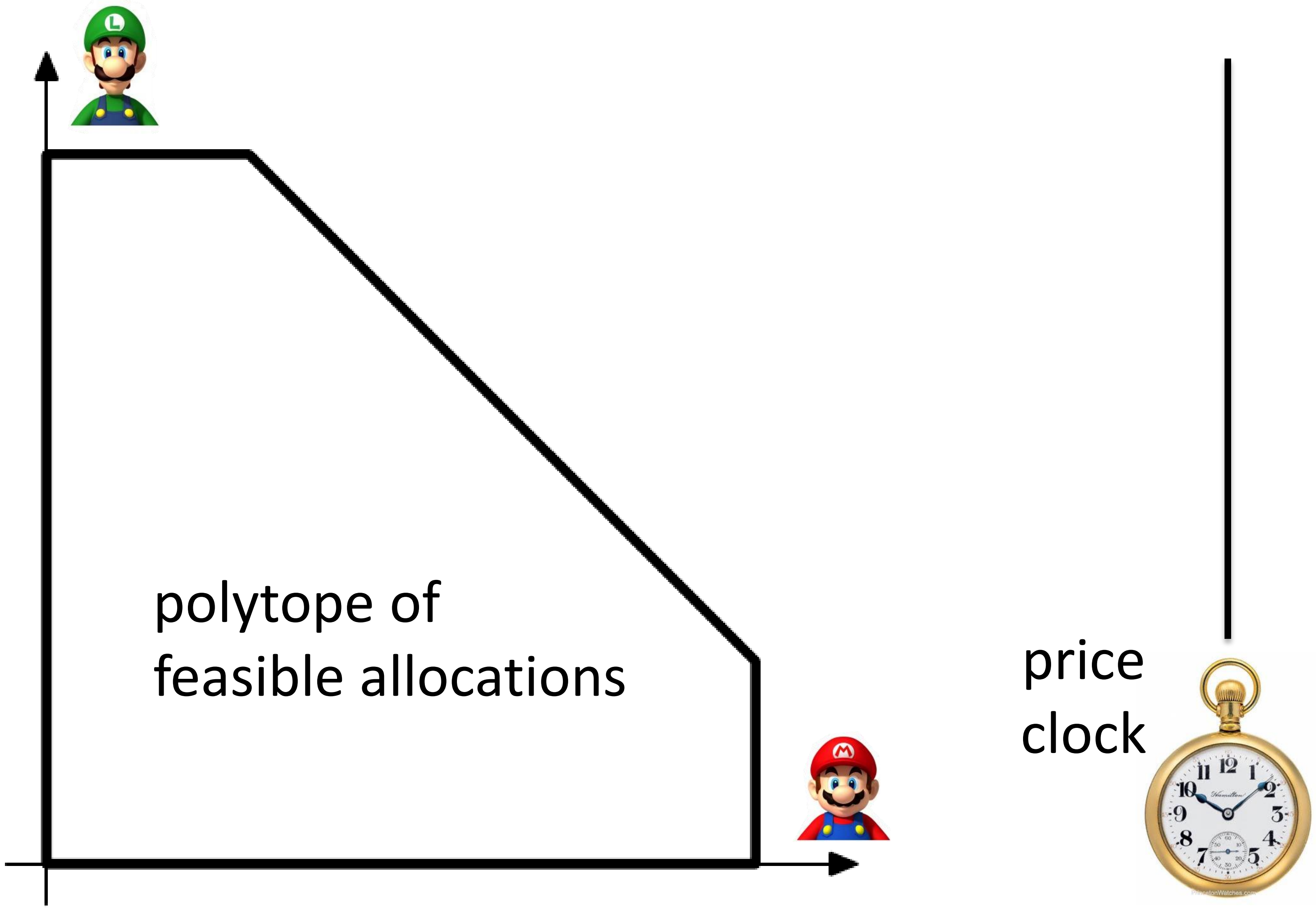
General model:

- multiple slots
- multiple keywords
- easy to generalize

Also on Sponsored Search with Budgets

Independently, [**Colini-Baldeschi, Henzinger, Leonardi, Starnberger**, 2012] design an auction for sponsored search with one keyword, multiple slots and budgets.

Our auction



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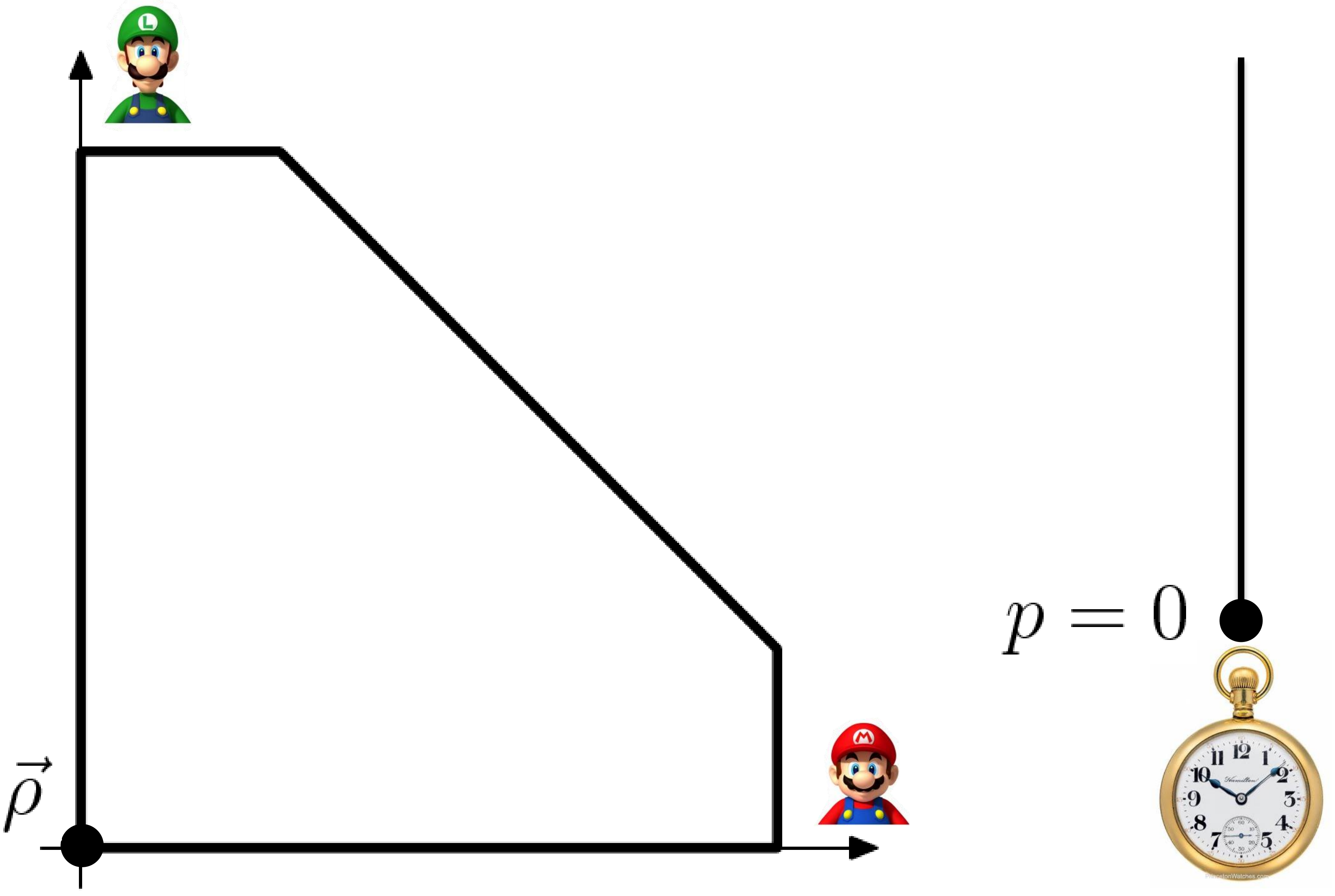

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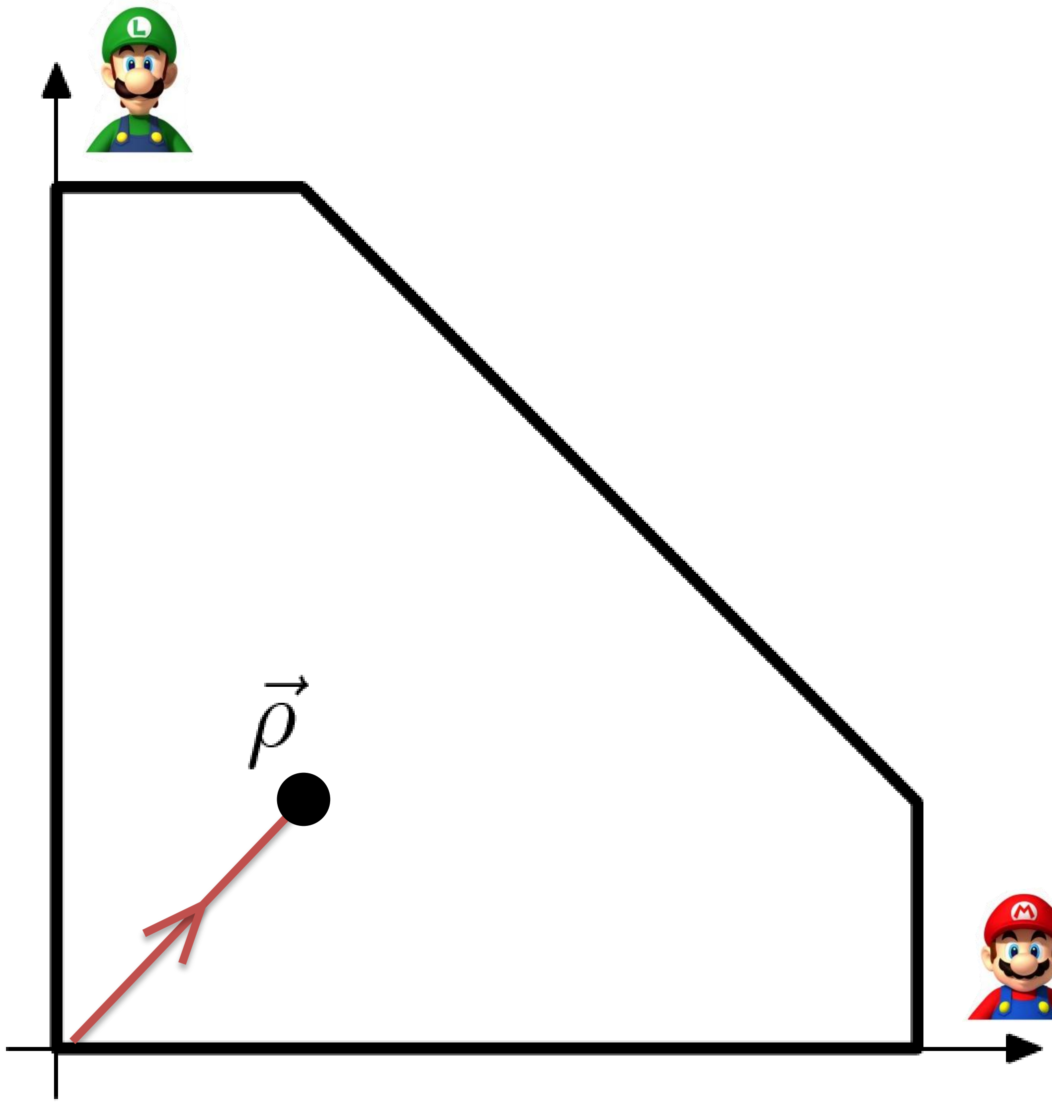

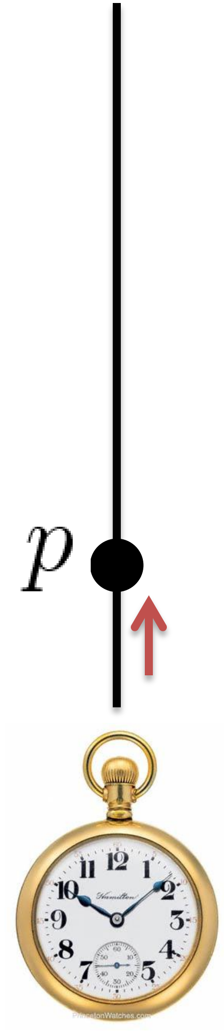

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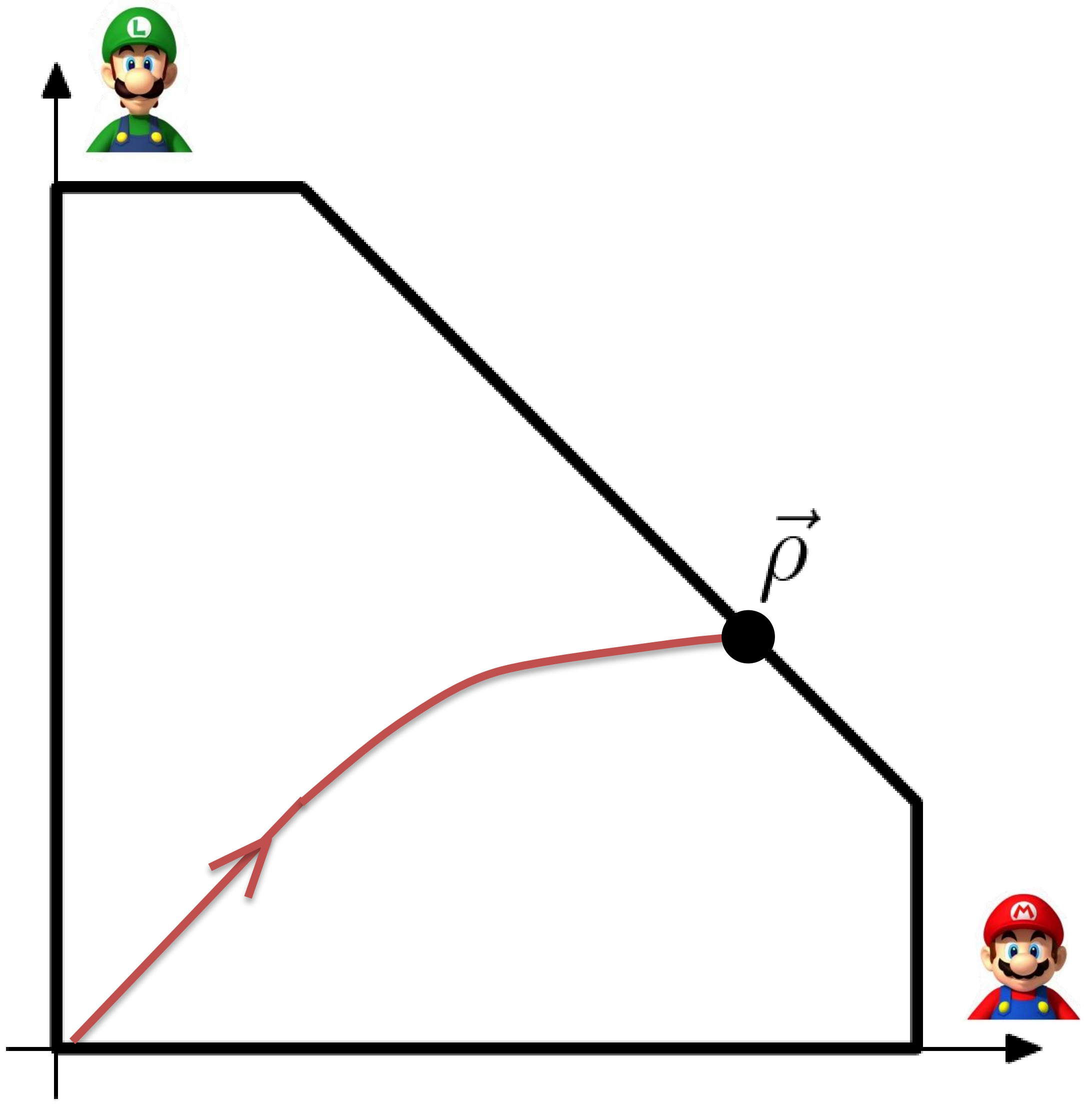

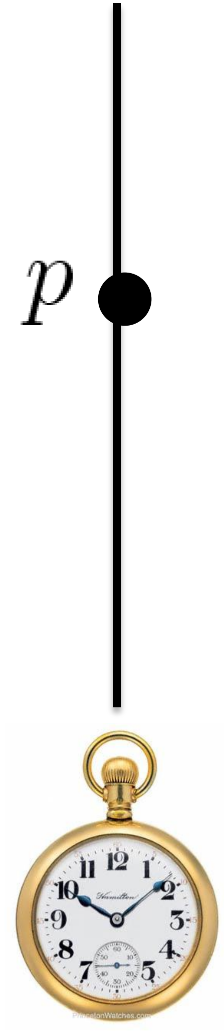

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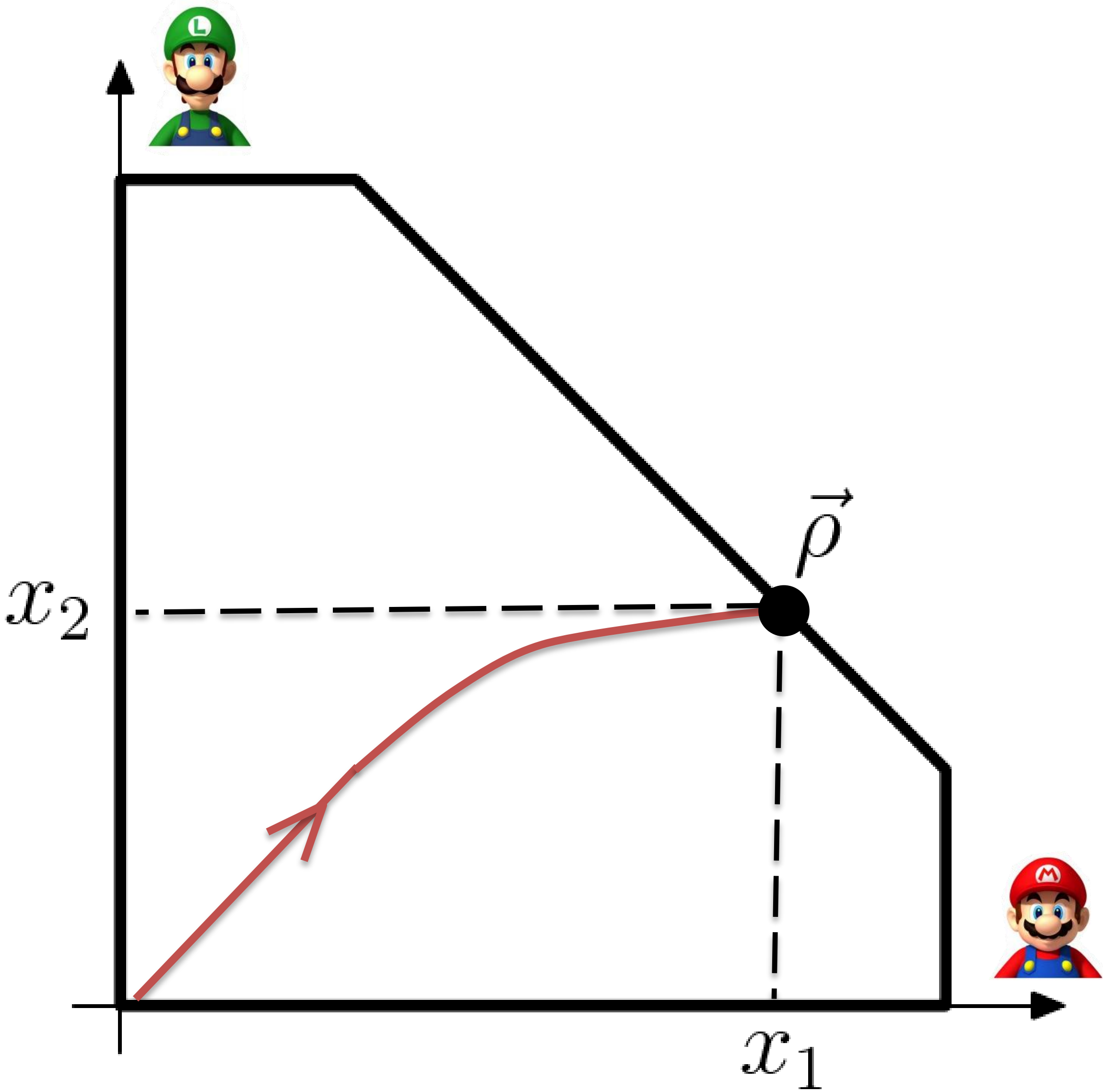

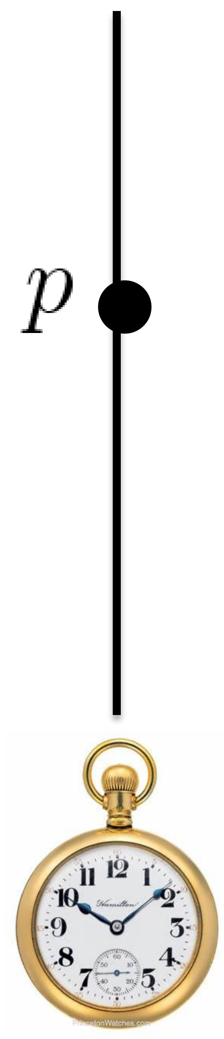

Our auction

In each step compute demands d_i at price p

$$d_i = \frac{B_i^{\text{rem}}}{p} \text{ if } p \leq v_i \text{; and } 0 \text{ o.w.}$$

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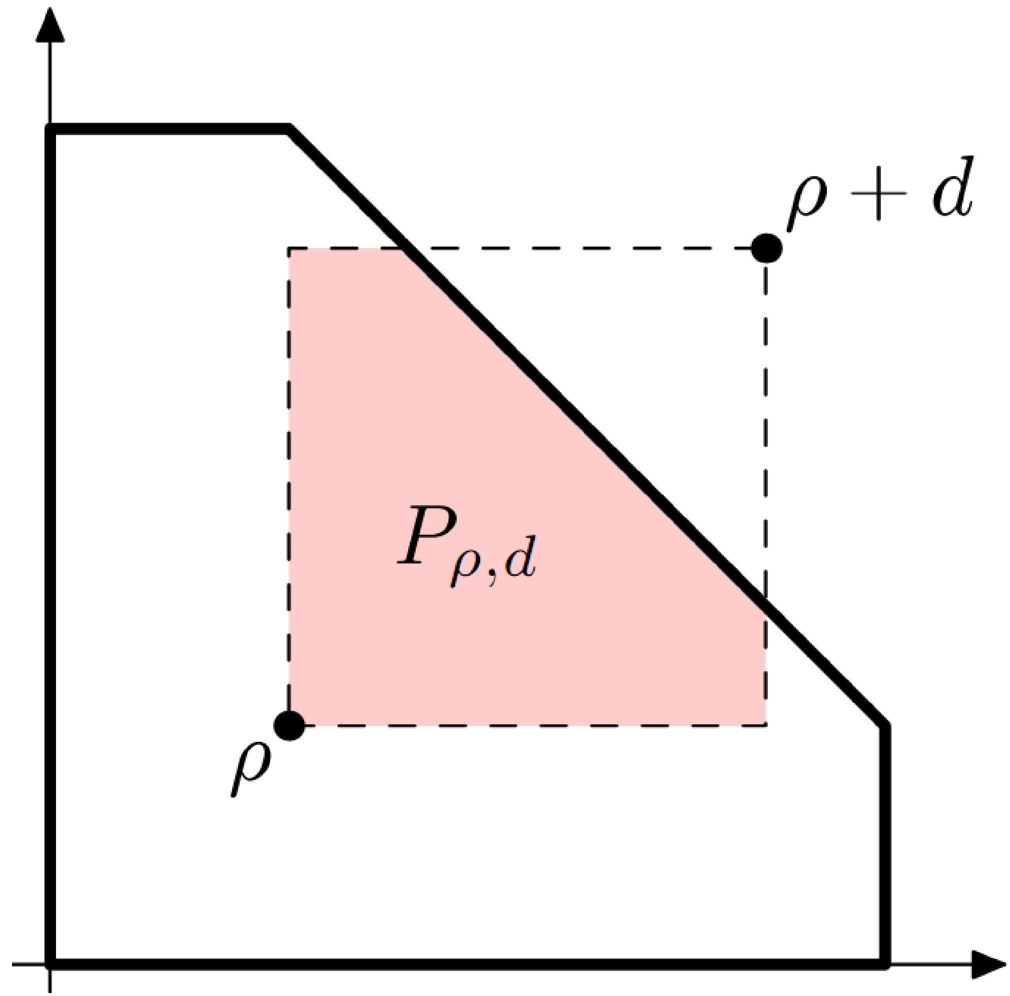
Compute **clinched** amount δ_i

$$\rho_i = \rho_i + \delta_i \qquad B_i = B_i - p\delta_i$$

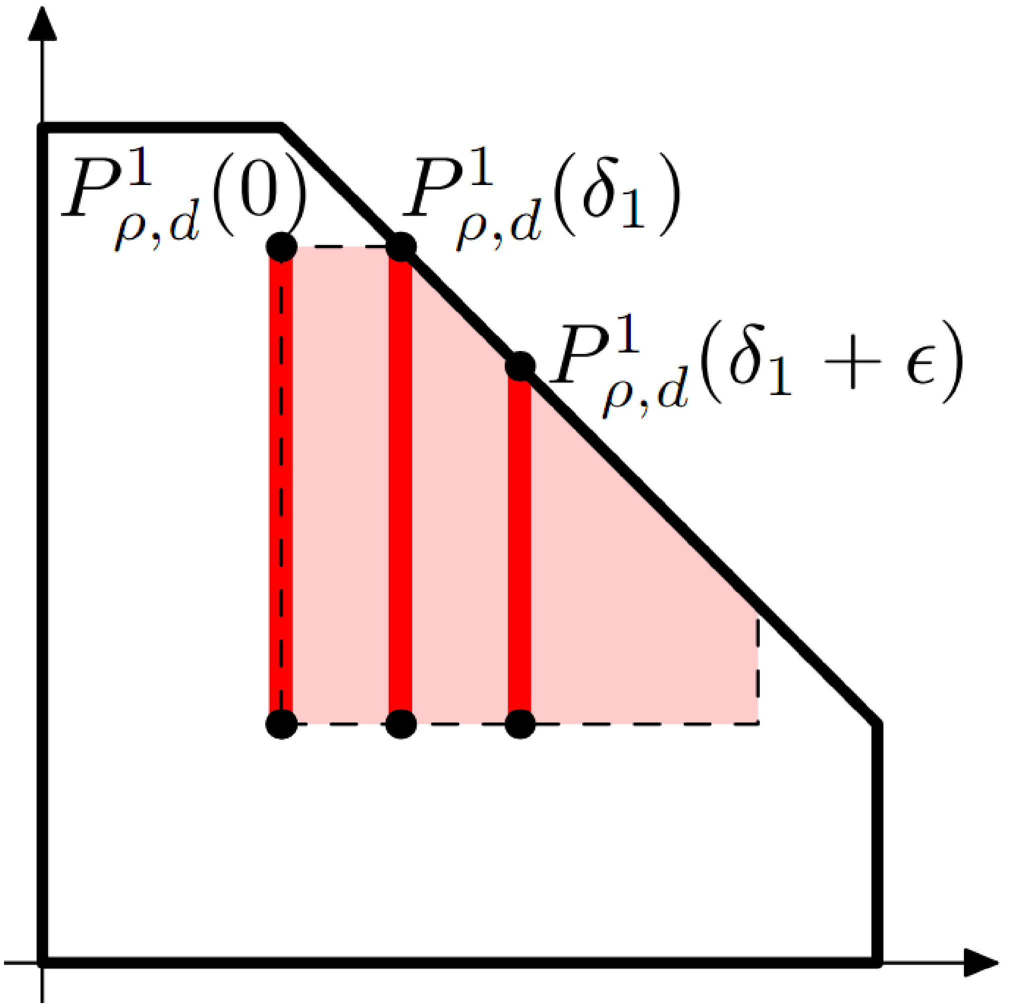
Computing clinched amounts δ_i

What is the allocations that are still feasible at this point?



Our auction: how to implement clinch ?

How much can
I allocate to 1
without harming
player 2?



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$$P^i_{\rho,d}(x_i) = \{x_{-i} \in \mathbb{R}_+^{[n]\setminus i}; (x_i, x_{-i}) \in P_{\rho,d}\}$$

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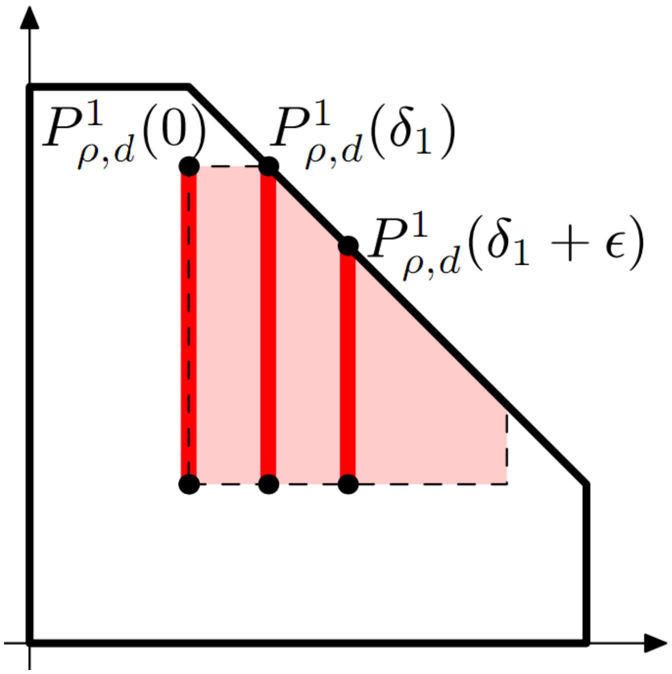

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Theorem: Clinching as defined above results in a feasible allocation. If **P** is a polymatroid, $\boldsymbol{\delta}_i$ can be computed efficiently using submodular minimization.

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[in practice there are more efficient algorithms for each case]

Summary of the proof

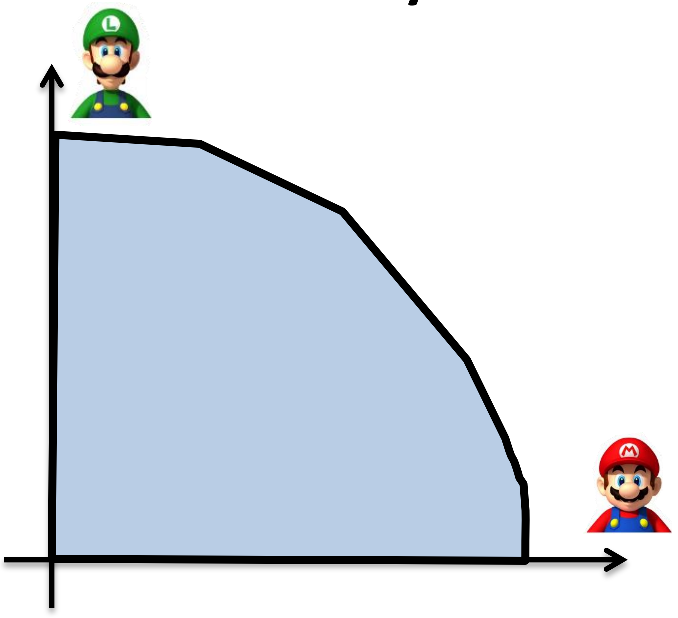
- Show clinching is well-defined and can be computed efficiently
- Characterize Pareto-optimal outcomes for polymatroidal environments
- Show that the auction produces an outcome satisfying the characterization

Extensions and Limits

Going beyond polymatroids…

General convex environment
One budget-constrained player

For a single budget constrained player (and many other unconstrained ones), it is possible do design an auction for any convex environment.



What about 2 budget constrained players ?

Weak impossibility: There is no auction following the clinching framework beyond (scaled) polymatroids.

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Weak impossibility: There is no auction following the clinching framework beyond (scaled) polymatroids.

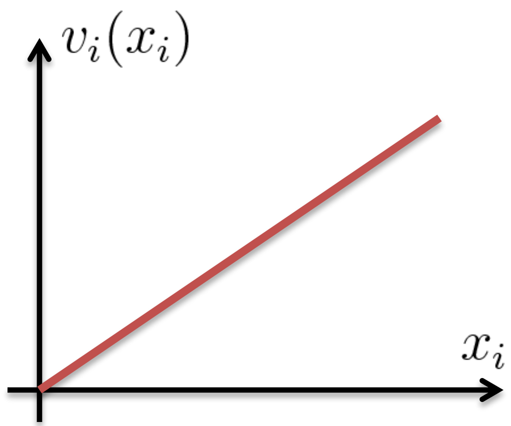
Stronger impossibility: There exists a class of polytopes, for which **no auction exists** satisfying all the desirable properties.

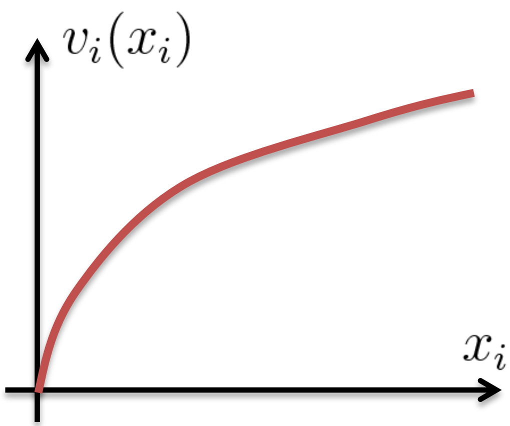
No hope of an auction for a general polyhedral environment.

Impossibility for decreasing marginals

Single divisible good: $\{x \in \mathbb{R}^n_+ ; \sum_i x_i \leq 1\}$

Decreasing marginal valuations





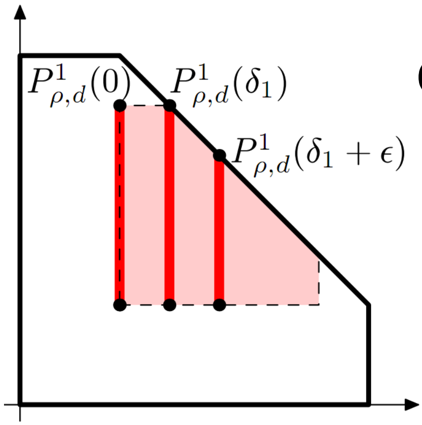
Impossibility for decreasing marginals

Single divisible good: $\{x \in \mathbb{R}^n_+; \sum_i x_i \leq 1\}$

Thm: No auction with all the desirable properties for one divisible good with decreasing marginals.

Strengthens previous impossibility results of [Lavi, May'11] and [Fiat et al'11]

Summary



Clinching auction for polymatroids

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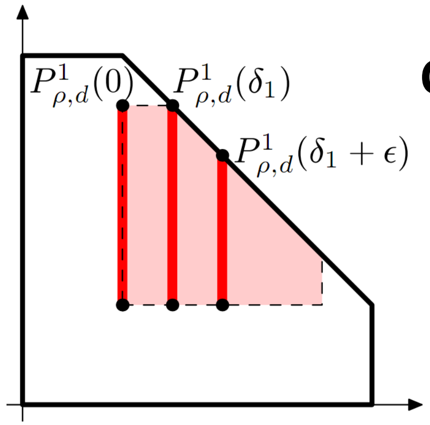

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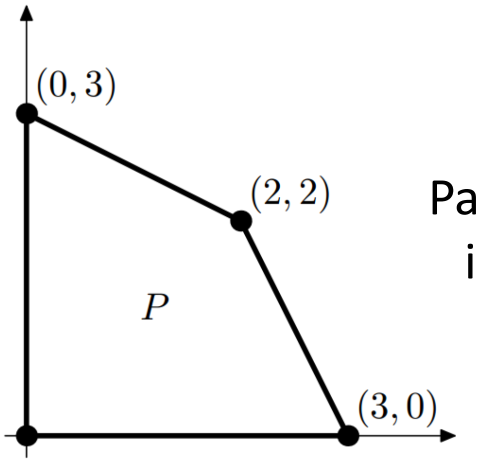
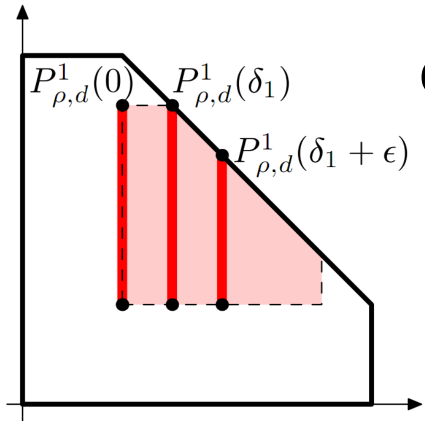


Characterization of Pareto Optimal Auctions in general polyhedral environments

Summary



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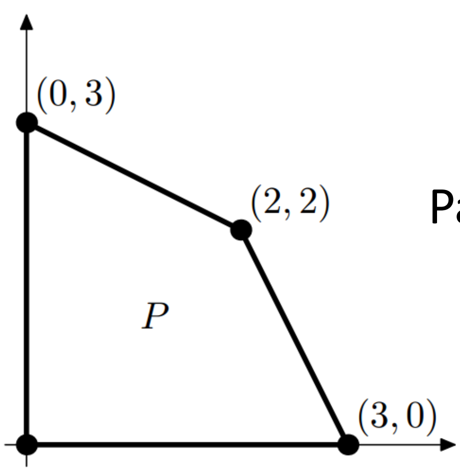

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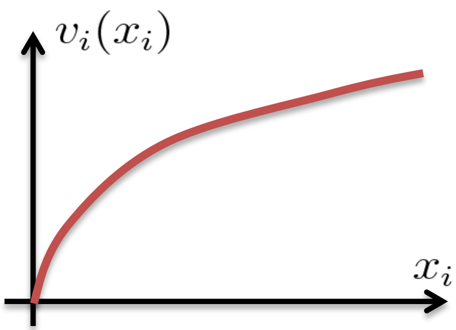
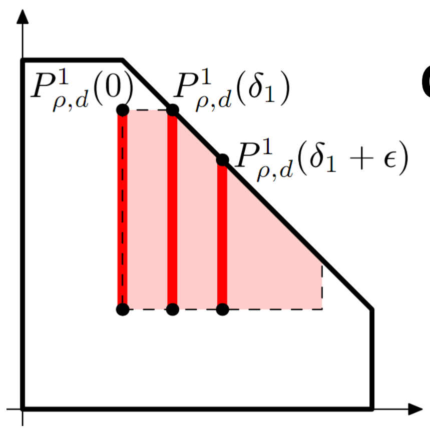
Impossibility for general polytopes



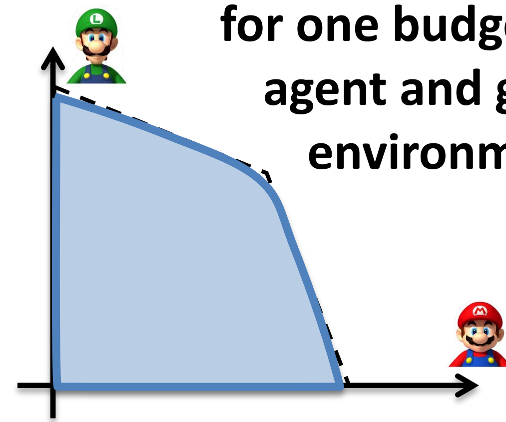
Impossibility for decreasing-marginals and budgets

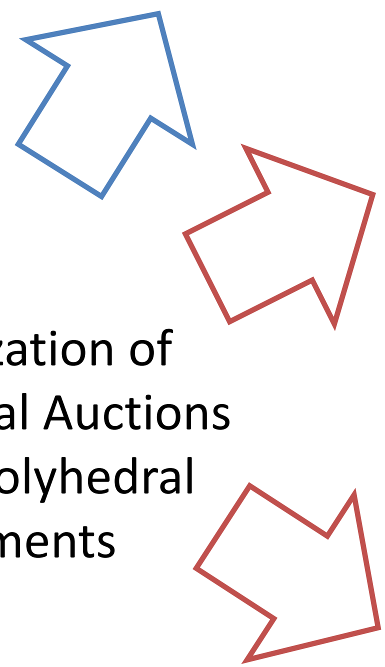
Summary



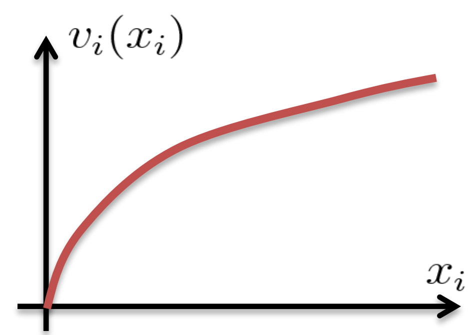
Clinching auction for polymatroids

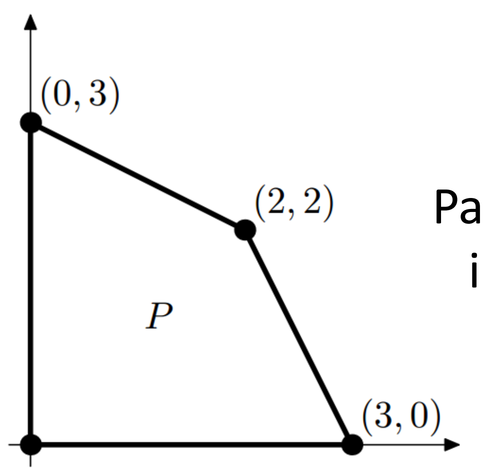
Positive results for one budget-constr agent and general environments





Impossibility for general polytopes





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Thanks !