

Bayes-Nash Price of Anarchy for GSP

Renato Paes Leme Éva Tardos

Cornell University

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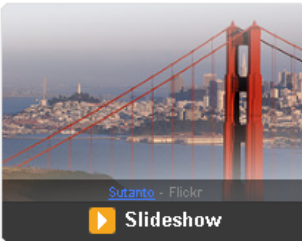
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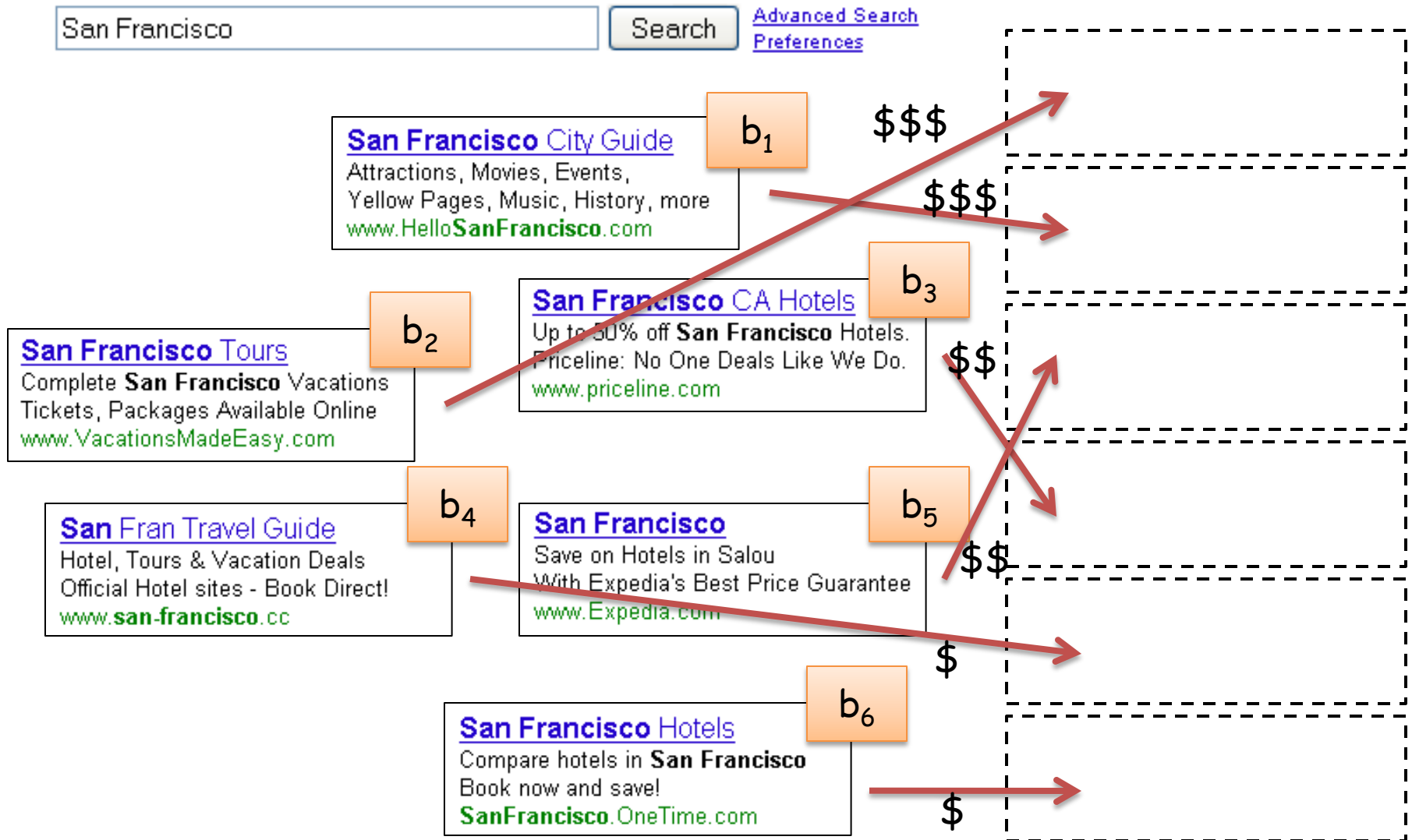
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Auction Model



Auction Model

San Francisco

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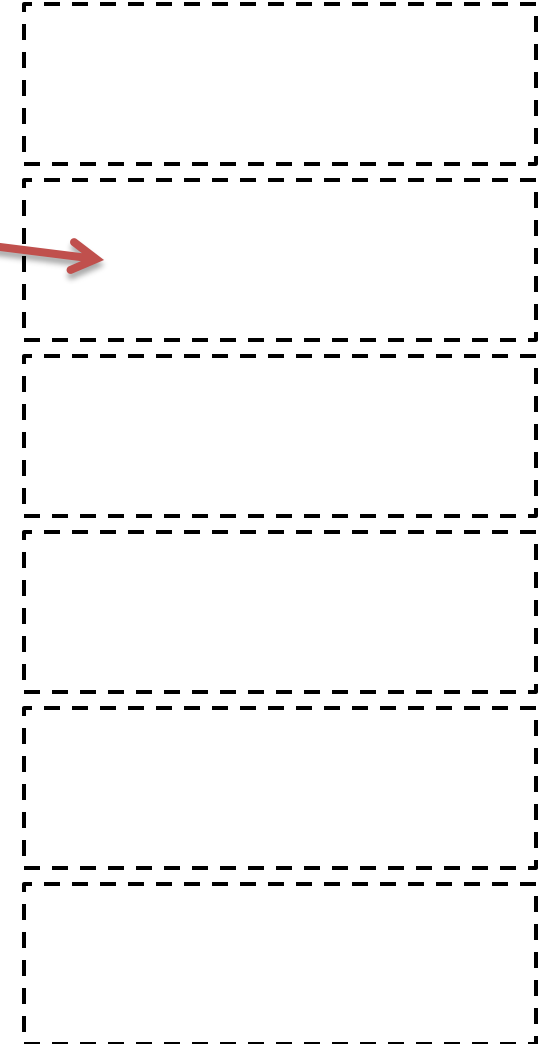
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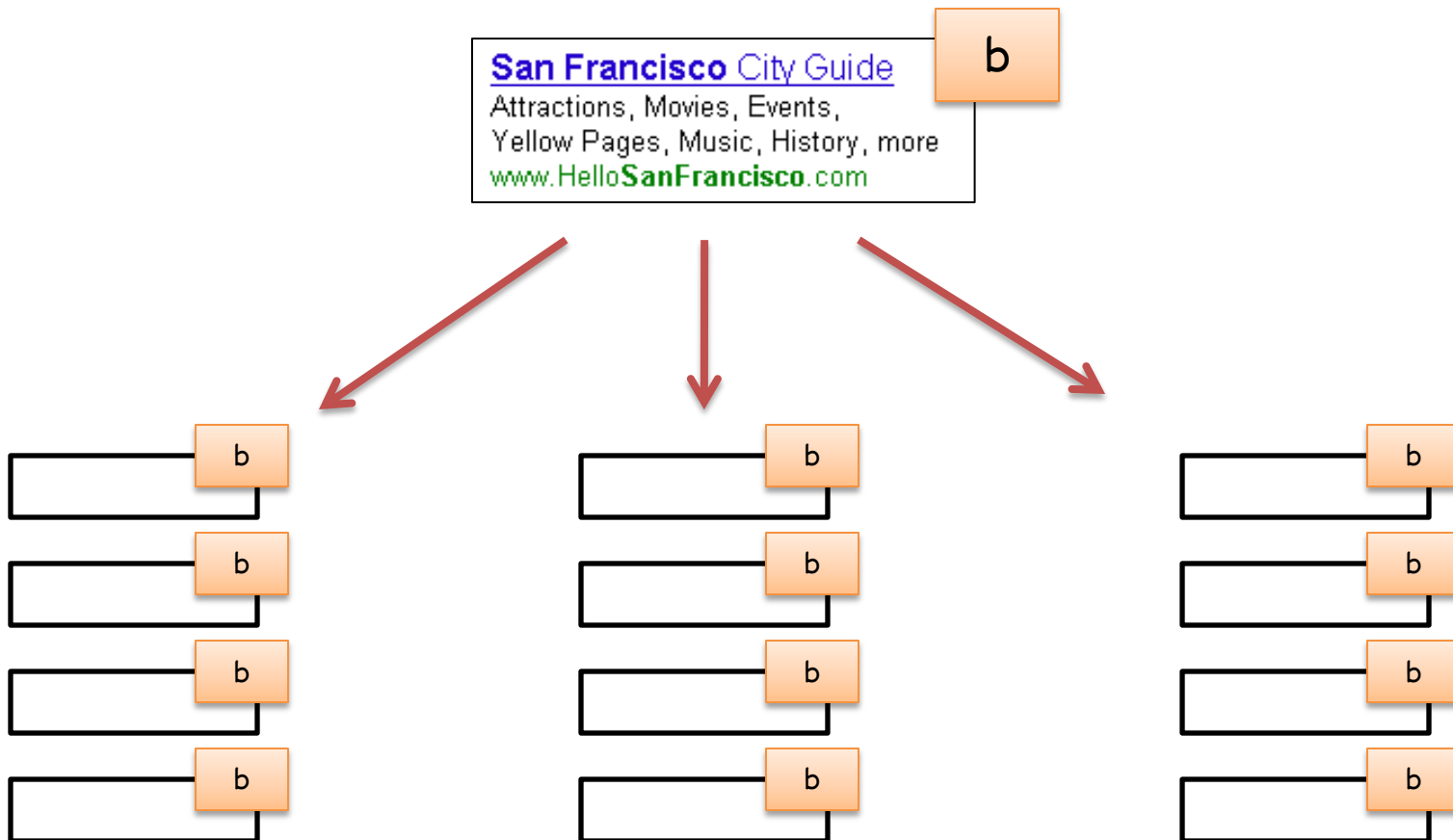


b?



Auction Model

Idea: Optimize against a **distribution**.



Bayes-Nash solution concept

- Bayes-Nash models the uncertainty of other players about valuations
- Values v_i are independent random vars
- Optimize against a distribution

Goal: bound the Bayes-Nash
Price of Anarchy

Bayes-Nash solution concept

- Bayes-Nash models the uncertainty of other players about valuations
- Values v_i are independent random vars
- Optimize against a distribution

Thm: Bayes-Nash PoA ≤ 8

Model

- n advertisers and n slots
- $v_i \sim V_i$ (valuations distribution)
- player i knows v_i and V_j for $j \neq i$
- Strategy: bidding function $b_i(v_i)$
- Assumption: $b_i(v_i) \leq v_i$

Model

$$v_1 \sim V_1$$



$$b_1(v_1)$$

$$v_2 \sim V_2$$

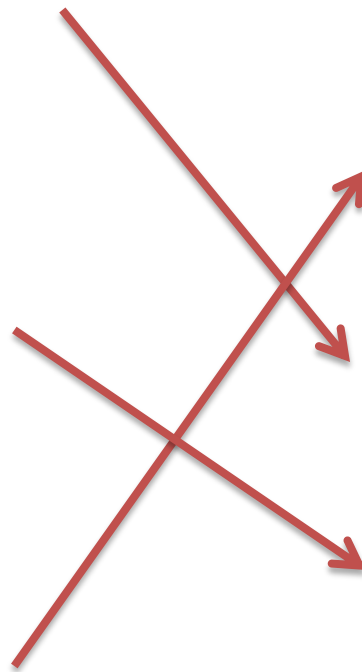


$$b_2(v_2)$$

$$v_3 \sim V_3$$



$$b_3(v_3)$$

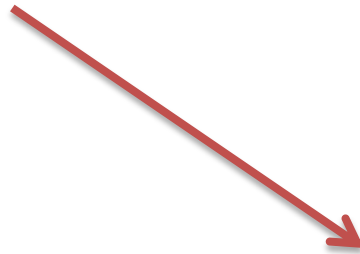
$$\alpha_1$$
$$\alpha_2$$
$$\alpha_3$$


Model

$$v_2 \sim V_2$$



$$b_2(v_2)$$



$$\alpha_3$$

Model

$$v_i \sim V_i$$



$$b_i(v_i)$$

$$\sigma = \pi^{-1}$$

$$\alpha_j$$

$$i = \pi(j)$$

$$j = \sigma(i)$$

Utility of player i :

$$u_i(b) = \alpha_{\sigma(i)} (v_i - b_{\pi(\sigma(i)) + 1})$$

Model

$$v_i \sim V_i$$



$$b_i(v_i)$$



$$\alpha_j$$

$$i = \pi(j)$$

$$j = \sigma(i)$$

Utility of player i :

$$u_i(b) = \alpha_{\sigma(i)} (v_i - b_{\pi(\sigma(i) + 1)})$$

next highest bid

Model

$$v_i \sim V_i$$



$$b_i(v_i)$$



$$\alpha_j$$

$$i = \pi(j)$$

$$j = \sigma(i)$$

Bayes-Nash equilibrium:

$$E[u_i(b_i, b_{-i}) | v_i] \geq E[u_i(b'_i, b_{-i}) | v_i]$$

Model

$$v_i \sim V_i$$



$$b_i(v_i)$$



$$\alpha_j$$

$$i = \pi(j)$$

$$j = \sigma(i)$$

Bayes-Nash equilibrium:

$$E[u_i(b_i, b_{-i}) | v_i] \geq E[u_i(b'_i, b_{-i}) | v_i]$$

Expectation over v_{-i}

Bayes-Nash Equilibrium

v_i are random variables

$\mu(i)$ = slot that player i occupies in
Opt (also a random variable)

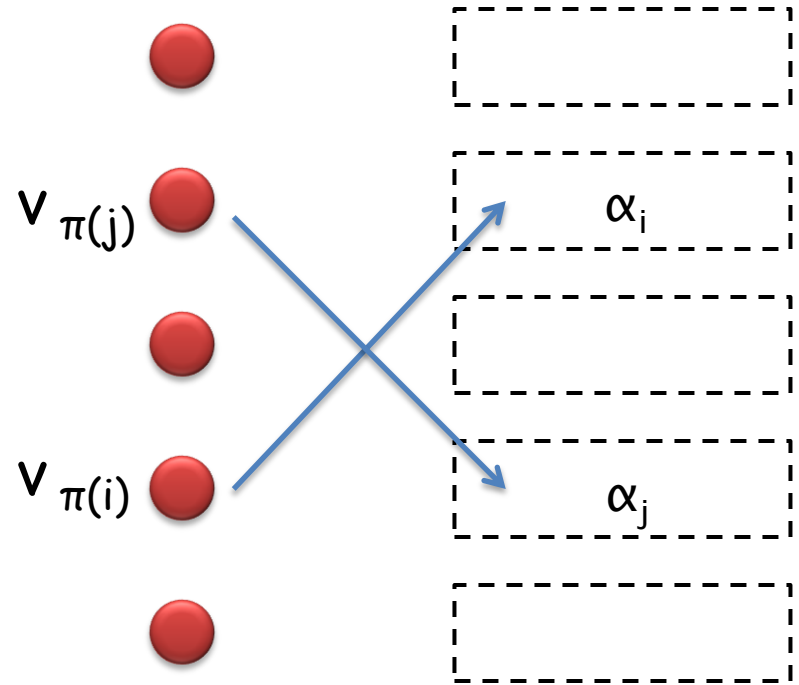
$$\text{Bayes-Nash PoA} = \frac{E[\sum_i v_i \alpha_{\mu(i)}]}{E[\sum_i v_i \alpha_{\sigma(i)}]}$$

Related results

- [PL-Tardos 09] prove a bound of 1.618 for (full information) PoA of GSP.
- [EOS][Varian] analyze full information setting
- [Gomes-Sweeney 09] study Bayes-Nash equilibria of GSP and characterize symmetric equilibria.

How was pure PoA proved?

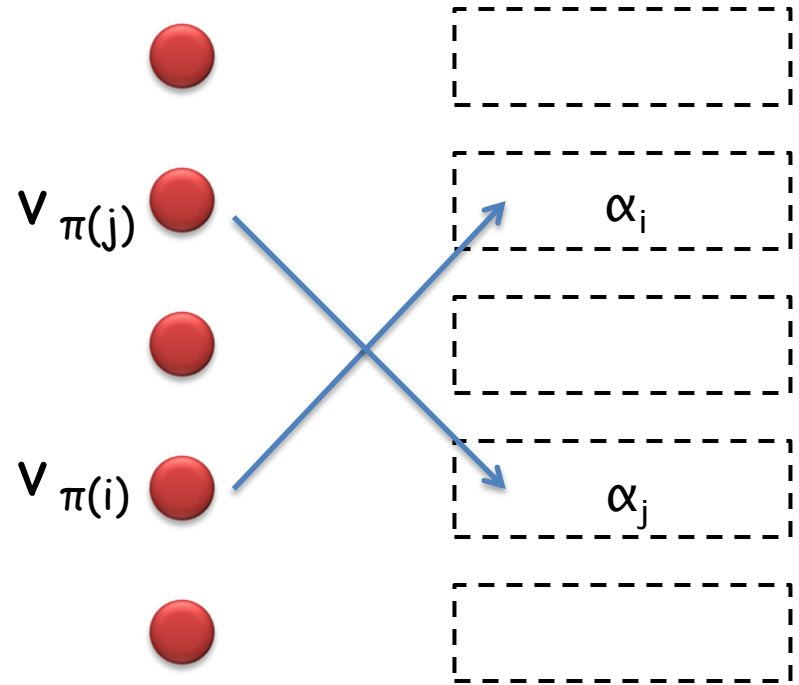
$$\frac{\alpha_j}{\alpha_i} + \frac{v_{\pi(i)}}{v_{\pi(j)}} \geq 1$$



- We need a structural characterization

How was pure PoA proved?

$$\alpha_j v_{\pi(j)} + \alpha_i v_{\pi(i)} \geq \alpha_i v_{\pi(j)}$$



- We need a structural characterization

New Structural Characterization

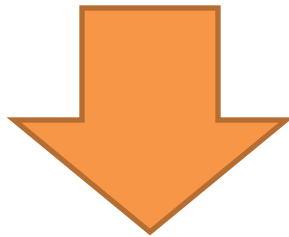
Lemma:

$$v_i E[\alpha_{\sigma(i)} | v_i] + E[\alpha_{\mu(i)} v_{\pi_{\mu}(i)} | v_i] \geq \frac{1}{4} v_i E[\alpha_{\mu(i)} | v_i]$$

New Structural Characterization

Lemma:

$$v_i E[\alpha_{\sigma(i)} | v_i] + E[\alpha_{\mu(i)} v_{\pi\mu(i)} | v_i] \geq \frac{1}{4} v_i E[\alpha_{\mu(i)} | v_i]$$



$$E[\sum_i v_i \alpha_{\pi(i)}] \geq (1/8) E[\sum_i v_i \alpha_{\mu(i)}]$$

Main Theorem

Lemma:

$$v_i E[\alpha_{\sigma(i)} | v_i] + E[\alpha_{\mu(i)} v_{\pi(\mu(i))} | v_i] \geq \frac{1}{4} v_i E[\alpha_{\mu(i)} | v_i]$$

Proof of main theorem:

$$\begin{aligned} SW &= (1/2) E[\sum_i \alpha_i v_{\pi(i)} + \alpha_{\sigma(i)} v_i] = \\ &= (1/2) E[\sum_i \alpha_{\mu(i)} v_{\pi(\mu(i))} + \alpha_{\sigma(i)} v_i] = \\ &= (1/2) E[\sum_i E[\alpha_{\mu(i)} v_{\pi(\mu(i))} | v_i] + v_i E[\alpha_{\sigma(i)} | v_i]] \\ &\geq (1/8) E[\sum_i v_i \alpha_{\mu(i)}] \end{aligned}$$

New Structural Characterization

Lemma:

$$v_i E[\alpha_{\sigma(i)} | v_i] + E[\alpha_{\mu(i)} v_{\pi_{\mu}(i)} | v_i] \geq \frac{1}{4} v_i E[\alpha_{\mu(i)} | v_i]$$

How to prove it ?

- Find the right deviation.
- But player i doesn't know his true slot
- Solution: try all slots

New Structural Characterization

Lemma:


$$v_i E[\alpha_{\sigma(i)} | v_i] + E[\alpha_{\mu(i)} v_{\pi_{\mu(i)}} | v_i] \geq \frac{1}{4} v_i E[\alpha_{\mu(i)} | v_i]$$

How to prove it ?

- Player i gets k or better if he bids $> b_{\pi(k)}^i$
- But this is a random variable ...
- Deviation bid: $2 E[b_{\pi(k)}^i | v_i, \mu(i) = k]$
- Gets slot k with $\frac{1}{2}$ probability (Markov)

New Structural Characterization

How to prove it ?

- also gets slot $j \leq k$ whenever $\mu(i) = j$: 
 $2 E[b_{\pi_i(k)}^i | v_i, \mu(i) = k]$ decreases with k
(here we use **independence**)
- Write Nash inequalities for those deviations:

$$v_i E[\alpha_{\sigma(i)} | v_i] \geq \sum_{j \geq k} \frac{1}{2} P(\mu(i)=k | v_i) \alpha_j (v_i - B_k), \forall k$$

New Structural Characterization

How to prove it ?

- Smart Dual averaging the expression:



$$v_i E[\alpha_{\sigma(i)} | v_i] \geq \sum_{j \geq k} \frac{1}{2} P(\mu(i)=k | v_i) \alpha_j (v_i - B_k), \forall k$$

- Maintain payments small and value large

Structural characterization:

$$v_i E[\alpha_{\sigma(i)} | v_i] + E[\alpha_{\mu(i)} v_{\pi_{\mu}(i)} | v_i] \geq \frac{1}{4} v_i E[\alpha_{\mu(i)} | v_i]$$

New Structural Characterization

How to prove it ?

- Dual averaging the expression:



$$v_i E[\alpha_{\sigma(i)} | v_i] \geq \sum_{j \geq k} \frac{1}{2} P(\mu(i)=k | v_i) \alpha_j (v_i - B_k)$$

- Maintain payments small and value large

Not a smoothness proof.

Conclusion

- Constant bound for Bayes-Nash PoA
- Uniform bounds **across all distributions**
- Future directions:
 - Improve the constant
 - Get rid of independence