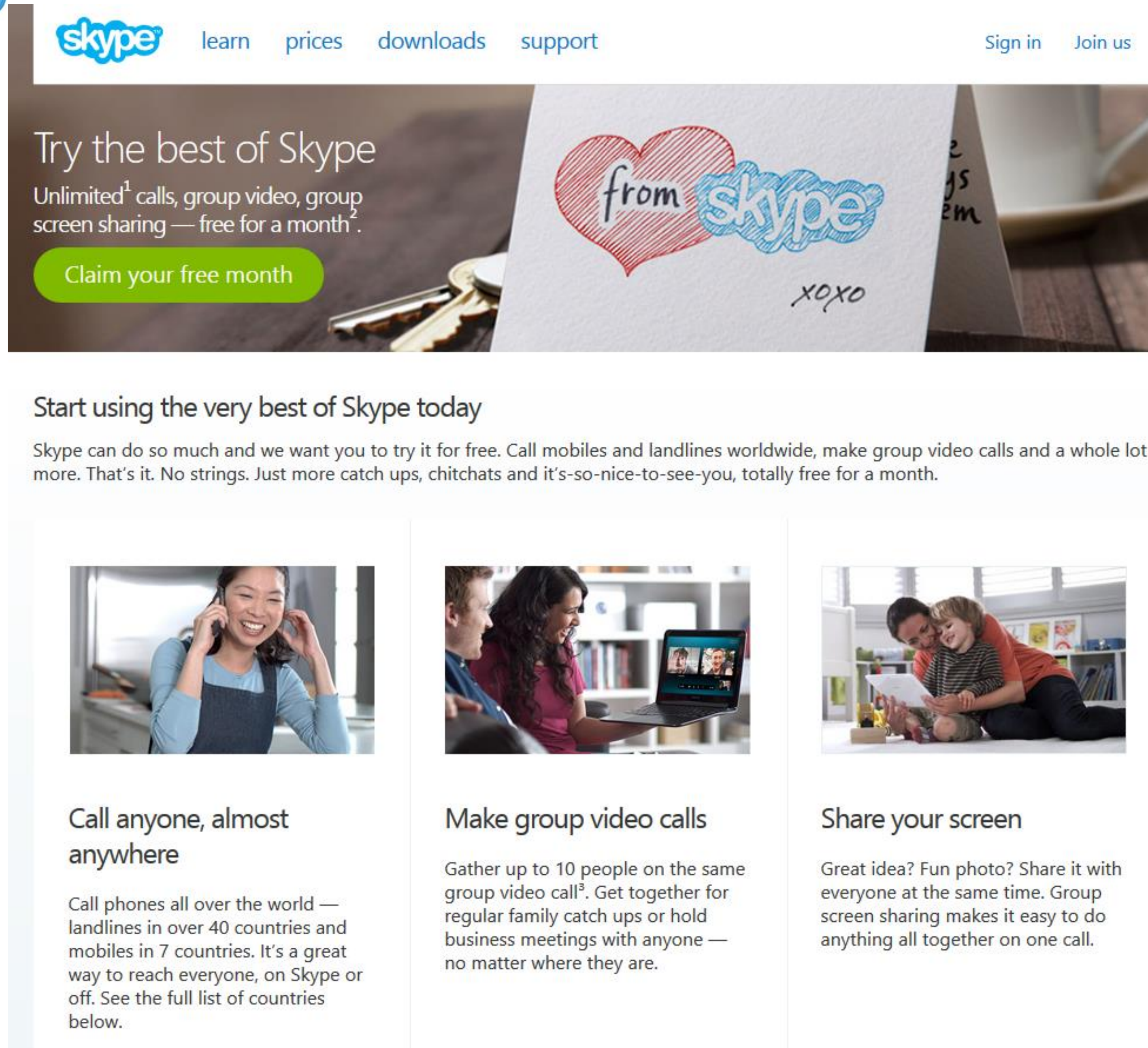
Pricing Public Goods for Private Sale

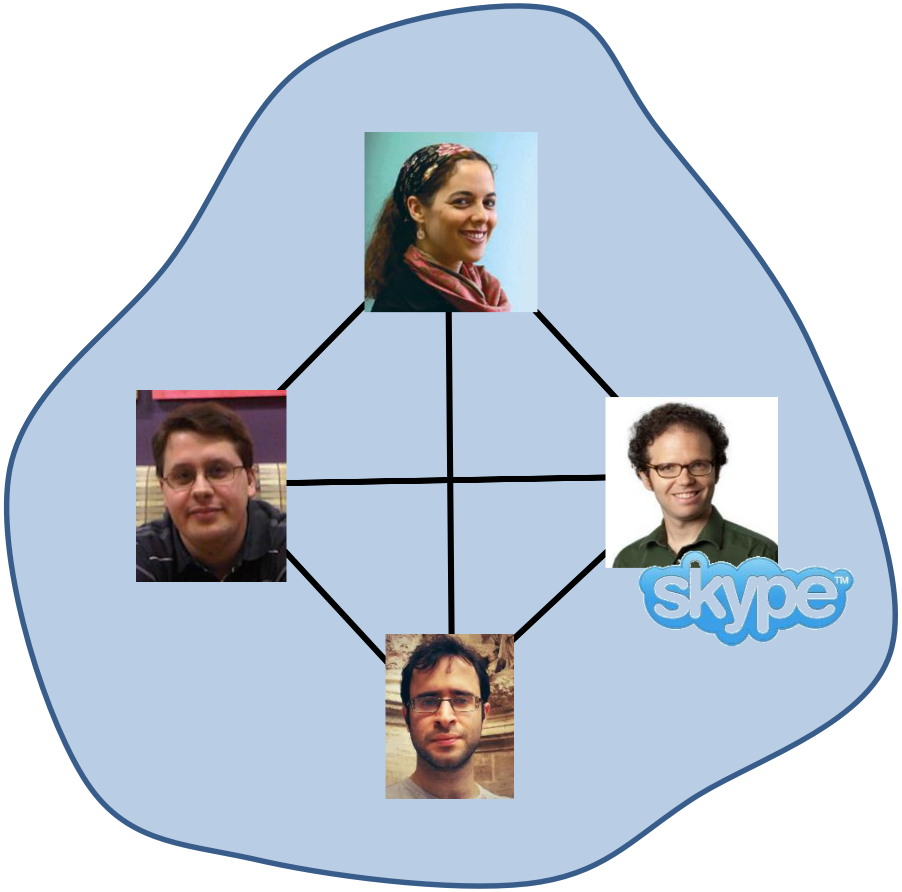
Michal Feldman
(Harvard and Hebrew U)

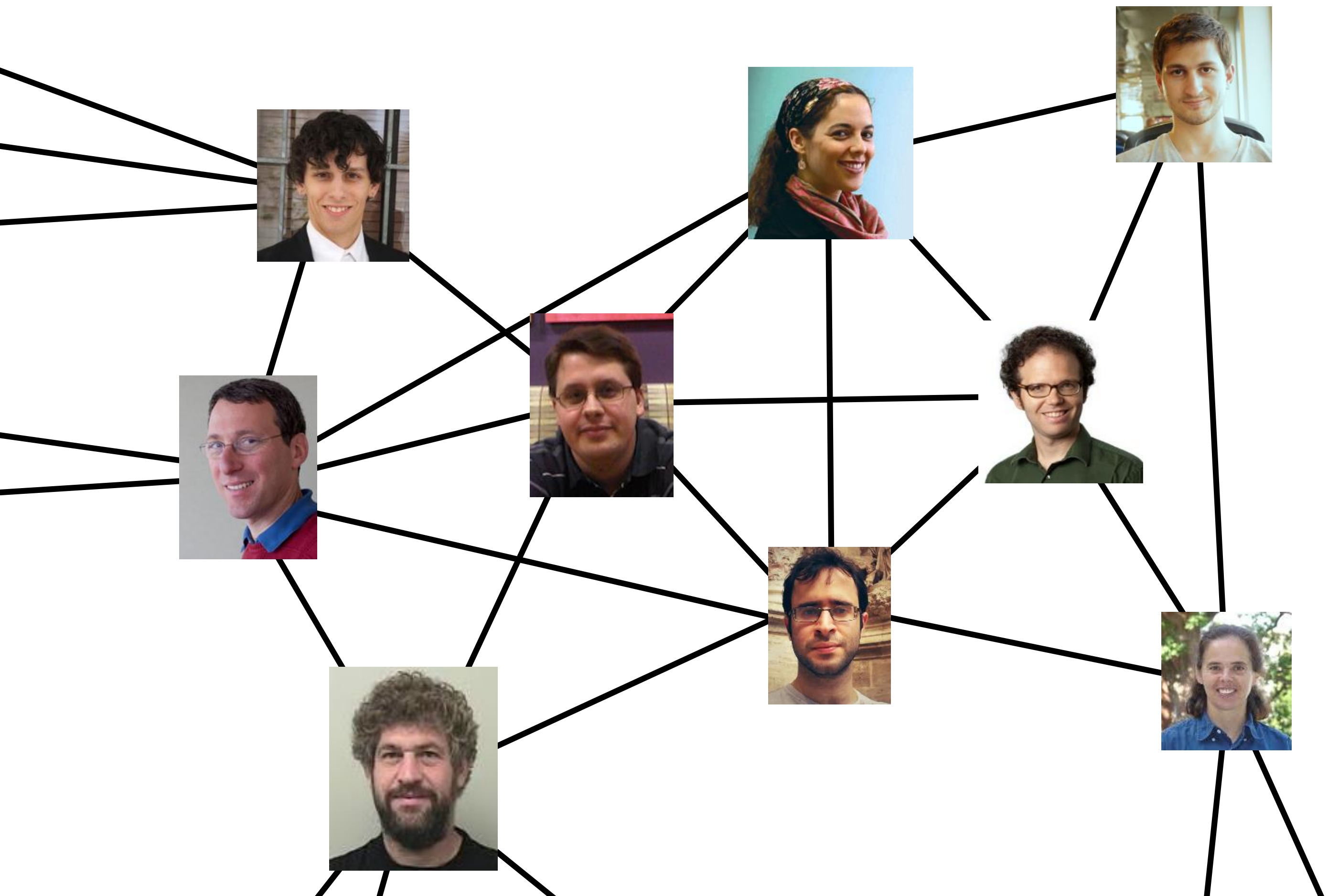
David Kempe
(U Southern California)

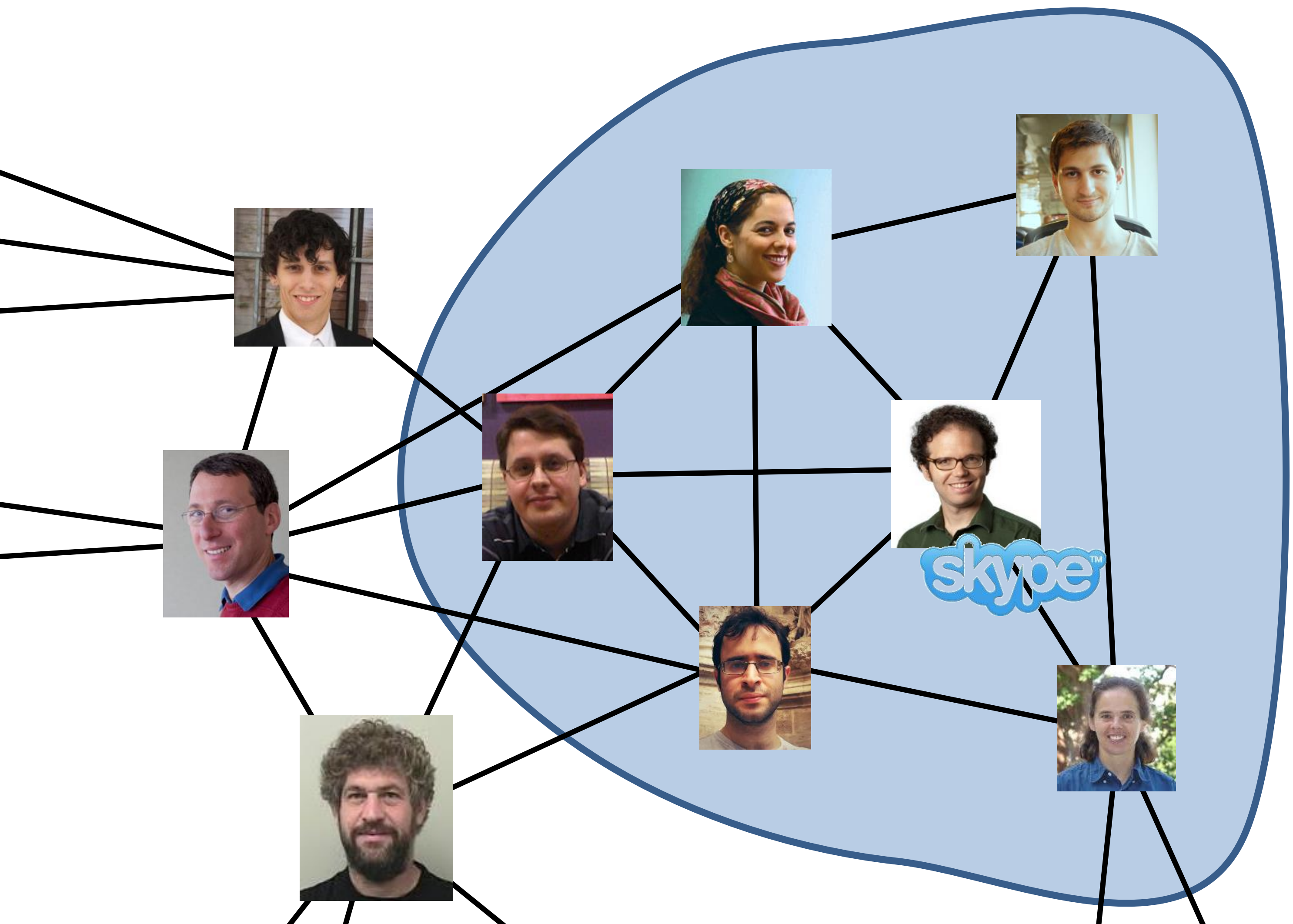
Brendan Lucier
(Microsoft Research)

Renato Paes Leme
(Microsoft Research)

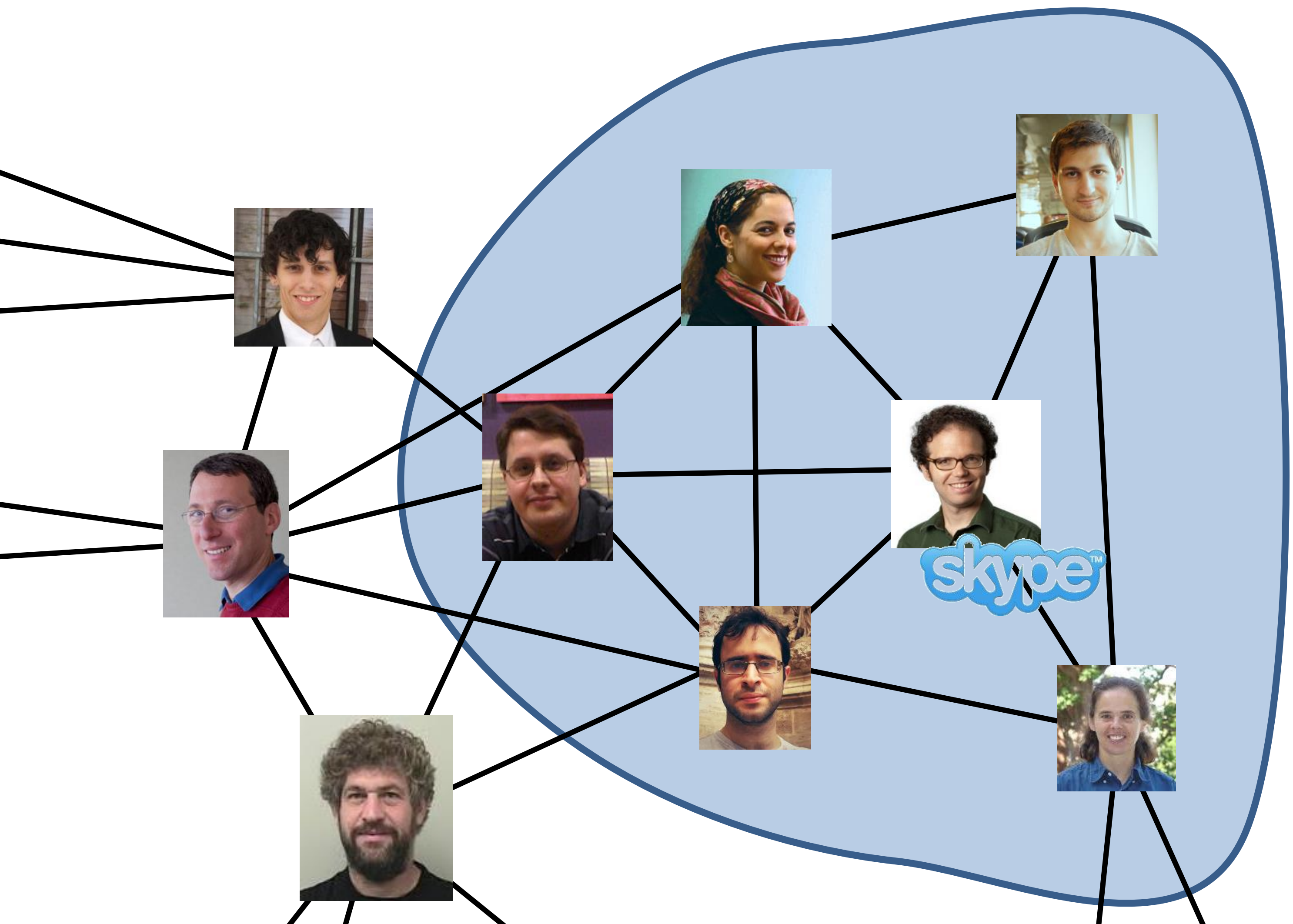

skype
learn
prices
downloads
support
Sign in
Join us
Try the best of Skype
Unlimited[1] calls, group video, group screen sharing — free for a month[2].
Claim your free month
from skype
xoxo
Start using the very best of Skype today
Skype can do so much and we want you to try it for free. Call mobiles and landlines worldwide, make group video calls and a whole lot more. That's it. No strings. Just more catch ups, chitchats and it's-so-nice-to-see-you, totally free for a month.
Call anyone, almost anywhere
Call phones all over the world — landlines in over 40 countries and mobiles in 7 countries. It's a great way to reach everyone, on Skype or off. See the full list of countries below.
Make group video calls
Gather up to 10 people on the same group video call[3]. Get together for regular family catch ups or hold business meetings with anyone — no matter where they are.
Share your screen
Great idea? Fun photo? Share it with everyone at the same time. Group screen sharing makes it easy to do anything all together on one call.


skype





Some products in social networks typically benefit not only the purchaser but also his friends (locally public goods).



Some products in social networks typically benefit not only the purchaser but also his friends (locally public goods).

- group video chat for Skype among collaborators
- poster printer among faculty in the department
- books among office mates
- snow-blower / gardening tools / ... among neighbors
- shared infrastructure (public wi-fi) among companies

Yet, most of those products are typically sold privately.

Our question: how to price goods over networks taking into account positive externalities ?

Related Work

- Public Goods: [Samuelson], [Bergstrom et al], ...
- Networked Public Goods: [Bramoullé, Kranton], [Bramoullé, Kranton, D'Amours], ...
- Negative Externalities: [Jehiel, Moldovanu], [Jehiel et al], [Brocas], ...
- Positive Externalities: [Hartline et al], [Arthur et al], [Akhalaghpour et al], [Anari et al], [Haghpanah et al], [Bhalgat et al]
- Pricing in Networks: [Candogan, Bimpikis, Ozdaglar], ...
- Pricing Public Goods: [Bergstrom, Blume, Varian], [Allouch], ...

Model of Locally Public Goods

- $[n]$ agents embedded in a social network $G = ([n], E)$
- agent i has value $v_i \sim F$ iid. Assume F is atomless
- utilities: if S is the set of allocated agents, then

$$u_i = v_i \cdot \mathbb{1}\{i \in S \vee S \cap N(i) \neq \emptyset\} - \pi_i$$

where $N(i)$ is the neighborhood of i.

Model of Locally Public Goods

- $[n]$ agents embedded in a social network $G = ([n], E)$
- agent i has value $v_i \sim F$ iid. Assume F is atomless
- utilities: if S is the set of allocated agents, then

$$u_i = v_i \cdot \mathbb{1}\{i \in S \vee S \cap N(i) \neq \emptyset\} - \pi_i$$

where $N(i)$ is the neighborhood of i .

Posted-prices game

- seller decides on prices p_i
- each agent learns his value v_i and the prices and decides whether to buy or not

$$v_i - p_i \geq v_i \cdot \mathbb{P}[\text{someone in } N(i) \text{ buys}]$$

Posted-prices game

- seller decides on prices p_i
- each agent learns his value v_i and the prices and decides whether to buy or not

$$v_i - p_i \geq v_i \cdot \mathbb{P}[\text{someone in } N(i) \text{ buys}]$$

Posted-prices game

- seller decides on prices p_i
- each agent learns his value v_i and the prices and decides whether to buy or not

$$v_i - p_i \geq v_i \cdot \mathbb{P}[\text{someone in } N(i) \text{ buys}]$$

$$v_i \geq \frac{p_i}{\prod_{j \in N(i)} \mathbb{P}[j \text{ does not buy}]} =: T_i$$

Posted-prices game

- seller decides on prices p_i
- each agent learns his value v_i and the prices and decides whether to buy or not

$$v_i - p_i \geq v_i \cdot \mathbb{P}[\text{someone in } N(i) \text{ buys}]$$

$$v_i \geq \frac{p_i}{\prod_{j \in N(i)} \mathbb{P}[j \text{ does not buy}]} =: T_i$$

$$\mathbb{P}[j \text{ does not buy}] = \mathbb{P}[v_j < T_j] = F(T_j)$$

Posted-prices game

- seller decides on prices p_i
- each agent learns his value v_i and the prices and decides whether to buy or not

$$v_i - p_i \geq v_i \cdot \mathbb{P}[\text{someone in } N(i) \text{ buys}]$$

$$v_i \geq \frac{p_i}{\prod_{j \in N(i)} \mathbb{P}[j \text{ does not buy}]} =: T_i$$

$$\mathbb{P}[j \text{ does not buy}] = \mathbb{P}[v_j < T_j] = F(T_j)$$

- equilibrium thresholds

$$T_i \cdot \prod_{j \in N(i)} F(T_j) = p_i, \forall i \in [n]$$

- revenue

$$\mathcal{R}(\mathbf{p}, \mathbf{T}) = \sum_i p_i (1 - F(T_i))$$

Posted-prices game

Lemma: For any prices $\mathbf{p}$ and distribution F , there is a vector of equilibrium thresholds. If $\mathbf{p}$ is uniform and the graph is regular, there is a symmetric equilibrium.

Proof by fixed point arguments.

Posted-prices game

Lemma: For any prices $\mathbf{p}$ and distribution F, there is a vector of equilibrium thresholds. If $\mathbf{p}$ is uniform and the graph is regular, there is a symmetric equilibrium.

Proof by fixed point arguments.

Goal: Find prices $\mathbf{p}$ to optimize/approximate the worst-case revenue over all equilibria, i.e., $\min_{\mathbf{T}\in\mathcal{N}_{\mathbf{p}}} \mathcal{R}(\mathbf{p}, \mathbf{T})$

Posted-prices game

Lemma: For any prices $\mathbf{p}$ and distribution F, there is a vector of equilibrium thresholds. If $\mathbf{p}$ is uniform and the graph is regular, there is a symmetric equilibrium.

Proof by fixed point arguments.

Goal: Find prices $\mathbf{p}$ to optimize/approximate the worst-case revenue over all equilibria, i.e., $\min_{\mathbf{T}\in\mathcal{N}_{\mathbf{p}}} \mathcal{R}(\mathbf{p},\mathbf{T})$

Questions:

1) What can we do if we have little or no knowledge of the network topology?

2) How does uniform (non-discriminatory) pricing perform ?

Three settings considered in our work

1) complete graph, regular distribution

2) d-regular graph, regular distribution

3) any graph, uniform distribution

Global Public Goods $G = K_n$

Theorem: For regular F and $G = K_n$, the uniform price

$$p = F^{-1}(1 - 1/n) \cdot (1 - 1/n)^{-1}$$

guarantees in the **worst** equilibrium a constant fraction (1/8) of the revenue of **any** equilibrium at **any** price vector.

Global Public Goods $G = K_n$

Theorem: For regular F and $G = K_n$, the uniform price

$$p = F^{-1}(1 - 1/n) \cdot (1 - 1/n)^{-1}$$

guarantees in the **worst** equilibrium a constant fraction (1/8) of the revenue of **any** equilibrium at **any** price vector.

price vector $\mathbf{p}$

asymmetric thresholds $\mathbf{T}$

symmetric threshold $T = F^{-1}(1 - \frac{1}{n})$

Global Public Goods $G = K_n$

Theorem: For regular F and $G = K_n$, the uniform price

$$p = F^{-1}(1 - 1/n) \cdot (1 - 1/n)^{-1}$$

guarantees in the **worst** equilibrium a constant fraction (1/8) of the revenue of **any** equilibrium at **any** price vector.

price vector $\mathbf{p}$

asymmetric thresholds $\mathbf{T}$

symmetric threshold $T = F^{-1}(1 - \frac{1}{n})$

1 similar revenue (factor 2)

Global Public Goods $G = K_n$

Theorem: For regular F and $G = K_n$, the uniform price

$$p = F^{-1}(1 - 1/n) \cdot (1 - 1/n)^{-1}$$

guarantees in the **worst** equilibrium a constant fraction (1/8) of the revenue of **any** equilibrium at **any** price vector.

price vector $\mathbf{p}$

asymmetric thresholds $\mathbf{T}$

symmetric threshold

$T = F^{-1}(1 - \frac{1}{n})$

2 $\geq \frac{1}{4}$ Myerson (1 item, n players)

1 similar revenue (factor 2)

Global Public Goods $G = K_n$

Theorem: For regular F and $G = K_n$, the uniform price

$$p = F^{-1}(1 - 1/n) \cdot (1 - 1/n)^{-1}$$

guarantees in the **worst** equilibrium a constant fraction (1/8) of the revenue of **any** equilibrium at **any** price vector.

price vector $\mathbf{p}$

asymmetric thresholds $\mathbf{T}$

symmetric threshold

$T = F^{-1}(1 - \frac{1}{n})$

2

$\geq \frac{1}{4}$ Myerson (1 item, n players)

$\geq$ posted prices T_i'

$\geq \mathcal{R}(\mathbf{p}', \mathbf{T}')$

3

1 similar revenue (factor 2)

Locally Public Goods (d-regular graphs)

Theorem: For regular F and d-regular graph G, the uniform price $p = F^{-1}(1 - 1/d) \cdot (1 - 1/d)^d$, guarantees in the **worst** equilibrium a constant fraction of the revenue of **worst** equilibrium at **any uniform** price.

Locally Public Goods (d-regular graphs)

Theorem: For regular F and d-regular graph G, the uniform price $p = F^{-1}(1-1/d) \cdot (1-1/d)^d$, guarantees in the **worst** equilibrium a constant fraction of the revenue of **worst** equilibrium at **any uniform** price.

Show that this is necessary:

- unbounded gap between best best-case and best worst-case revenue
- unbounded gap between discriminatory and non-discriminatory pricing

Proof uses the prophet price rather then the Myerson price.

Locally Public Goods (any graph)

Theorem: For [0,1]-uniform F and generic G the uniform price $p = \frac{1}{2}$ guarantees a $\frac{4}{e}$ fraction of the revenue of **worst** equilibrium at **any uniform** price.

Theorem: For uniform F and generic G, approximating the optimal revenue within a $O(n^{1-\epsilon})$ factor is NP-hard.

I.e., we know a price that guarantees good revenue, yet knowing this value is hard.

Open Questions and Future Directions

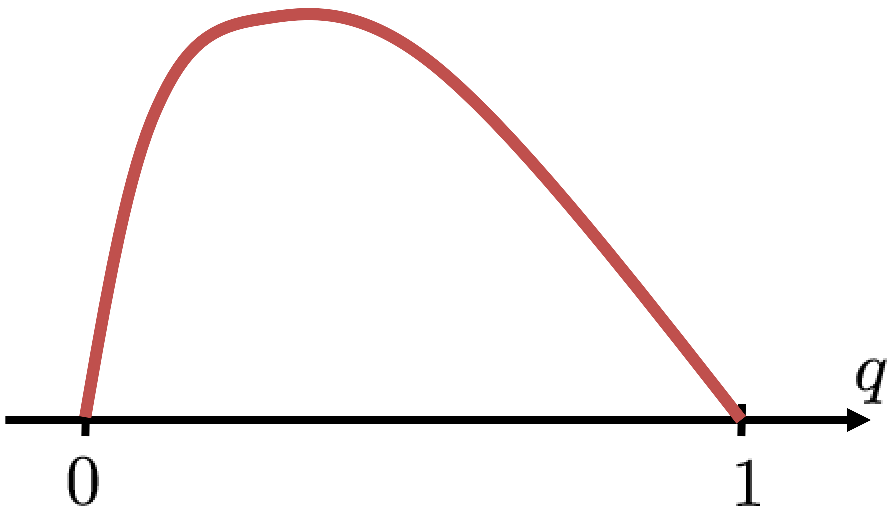
- Imperfect substitutes
- Strength of social ties (weighted edges)
- Non-identical / Non-regular distributions
- Other objective functions

Open Questions and Future Directions

- Imperfect substitutes
- Strength of social ties (weighted edges)
- Non-identical / Non-regular distributions
- Other objective functions

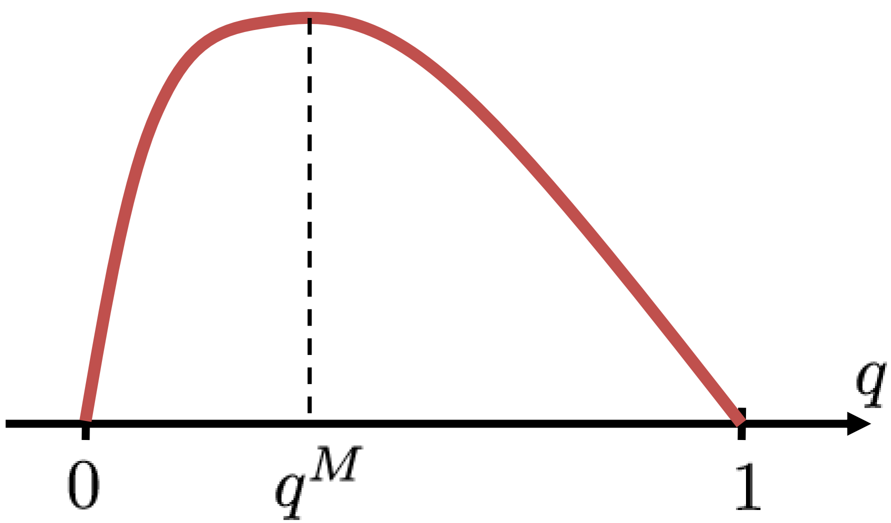
Thanks !

Regular Distributions



F is regular iff
$R(q) = q \cdot F^{-1}(1-q)$
is concave.

Regular Distributions



F is regular iff
$R(q) = q \cdot F^{-1}(1-q)$
is concave.

Myerson Price: p^M s.t. $1 - F(p^M) = q^M$

Virtual value: $\phi(v) = R'(1 - F(v))$

Global Public Goods $G = K_n$

Theorem: For regular F and $G = K_n$, the uniform price

$$p = F^{-1}(1 - 1/n) \cdot (1 - 1/n)^{-1}$$

guarantees in the **worst** equilibrium a constant fraction (1/8) of the revenue of **any** equilibrium at **any** price vector.

Proof: Inspired by a technique of [Chawla, Hartline, Kleinberg], we will compare the revenue with a posted price mechanism.

Given $\mathbf{p}, \mathbf{T} \in \mathcal{N}_{\mathbf{p}}$ we know that $p_i = T_i \cdot \prod_{j \neq i} F(T_j)$

$$\mathcal{R}(\mathbf{p}, \mathbf{T}) = \sum_i p_i (1 - F(T_i)) = \sum_i T_i (1 - F(T_i)) \prod_{j \neq i} F(T_i) \leq$$

$$\leq \sum_i T_i (1 - F(T_i)) \prod_{j < i} F(T_i) \leq \mathcal{R}^M$$

Global Public Goods $G = K_n$

Now, consider p as above and the corresponding symmetric equilibrium, $T_i = F^{-1}(1 - 1/n)$

Case 1. $T > p^M$ $\qquad \nu = \phi(T) > 0$

$$\begin{aligned} \mathcal{R}^M &= \mathbb{E}[\max_i \phi(v_i)^+] \leq \\ &\leq \nu \cdot \mathbb{P}[0 \leq \phi(v_i) \leq \nu] + \mathbb{E}[\max_i \phi(v_i) \mathbb{1}\{\phi(v_i) > \nu\}] \end{aligned}$$

Global Public Goods $G = K_n$

Now, consider p as above and the corresponding symmetric equilibrium, $T_i = F^{-1}(1 - 1/n)$

Case 1. $T > p^M$ $\nu = \phi(T) > 0$

$$\mathcal{R}^M = \mathbb{E}[\max_i \phi(v_i)^+] \leq$$

$$\leq \nu \cdot \boxed{\mathbb{P}[0 \leq \phi(v_i) \leq \nu]} + \mathbb{E}[\max_i \phi(v_i) \mathbb{1}\{\phi(v_i) > \nu\}]$$

$$\leq \mathbb{P}[\max_i v_i \leq T]$$

$$\leq (1 - F(T))^n =$$

$$= (1 - 1/n)^n \leq 1/e$$

Global Public Goods $G = K_n$

Now, consider p as above and the corresponding symmetric equilibrium, $T_i = F^{-1}(1 - 1/n)$

Case 1. $T > p^M$ $\quad \nu = \phi(T) > 0$

$$\mathcal{R}^M = \mathbb{E}[\max_i \phi(v_i)^+] \leq$$

$$\leq \nu \cdot \mathbb{P}[0 \leq \phi(v_i) \leq \nu] + \mathbb{E}[\max_i \phi(v_i) \mathbb{1}\{\phi(v_i) > \nu\}]$$

$$\leq \mathbb{P}[\max_i v_i \leq T]$$
$$\leq (1 - F(T))^n =$$
$$= (1 - 1/n)^n \leq 1/e$$

$$\leq \sum_i \mathbb{E}[\phi(v_i) \cdot \mathbb{1}\{\phi(v_i) > \nu\}] =$$
$$\leq \sum_i T(1 - F(T)) = T$$

Global Public Goods $G = K_n$

Now, consider p as above and the corresponding symmetric equilibrium, $T_i = F^{-1}(1 - 1/n)$

Case 1. $T > p^M$ $\quad \nu = \phi(T) > 0$

$$\mathcal{R}^M = \mathbb{E}[\max_i \phi(v_i)^+] \leq$$

$$\leq \nu \cdot \mathbb{P}[0 \leq \phi(v_i) \leq \nu] + \mathbb{E}[\max_i \phi(v_i) \mathbb{1}\{\phi(v_i) > \nu\}]$$

$$\nu = \phi(T) \leq T$$

$$\leq \mathbb{P}[\max_i v_i \leq T]$$
$$\leq (1 - F(T))^n =$$
$$= (1 - 1/n)^n \leq 1/e$$

$$\leq \sum_i \mathbb{E}[\phi(v_i) \cdot \mathbb{1}\{\phi(v_i) > \nu\}] =$$
$$\leq \sum_i T(1 - F(T)) = T$$

Global Public Goods $G = K_n$

Now, consider p as above and the corresponding symmetric equilibrium, $T_i = F^{-1}(1 - 1/n)$

Case 1. $T > p^M \qquad \nu = \phi(T) > 0$

$$
\begin{aligned}
\mathcal{R}^M &= \mathbb{E}[\max_i \phi(v_i)^+] \leq \\
&\leq \nu \cdot \mathbb{P}[0 \leq \phi(v_i) \leq \nu] + \mathbb{E}[\max_i \phi(v_i) \mathbb{1}\{\phi(v_i) > \nu\}] \\
&\leq (1 + 1/e) \cdot T
\end{aligned}
$$

Global Public Goods $G = K_n$

Now, consider p as above and the corresponding symmetric equilibrium, $T_i = F^{-1}(1 - 1/n)$

Case 1. $T > p^M$ $\qquad \nu = \phi(T) > 0$

$$\begin{aligned}\mathcal{R}^M &= \mathbb{E}[\max_i \phi(v_i)^+] \leq \\ &\leq \nu \cdot \mathbb{P}[0 \leq \phi(v_i) \leq \nu] + \mathbb{E}[\max_i \phi(v_i) \mathbb{1}\{\phi(v_i) > \nu\}] \\ &\leq (1 + 1/e) \cdot T\end{aligned}$$

$$\mathcal{R}(p, T) = \textstyle\sum_i T F(T)^{n-1}(1 - F(T)) = T(1 - 1/n)^{n-1} \geq T/e$$

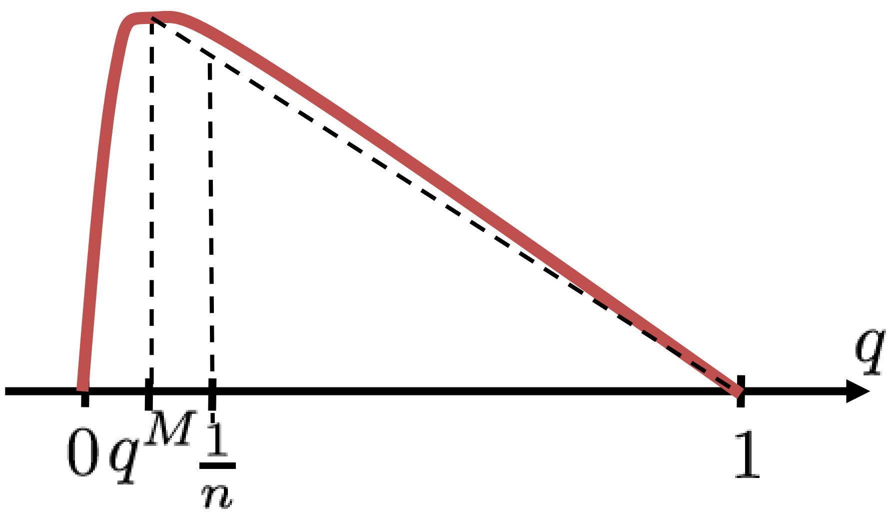
$$\mathcal{R}(p, T) \geq \mathcal{R}^M/(1 + e)$$

Global Public Goods $G = K_n$

Now, consider p as above and the corresponding symmetric equilibrium, $T_i = F^{-1}(1 - 1/n)$

Case 2. $T \leq p^M$

$$T(1 - F(T)) \geq (1 - 1/n)p^M(1 - F(p^M))$$



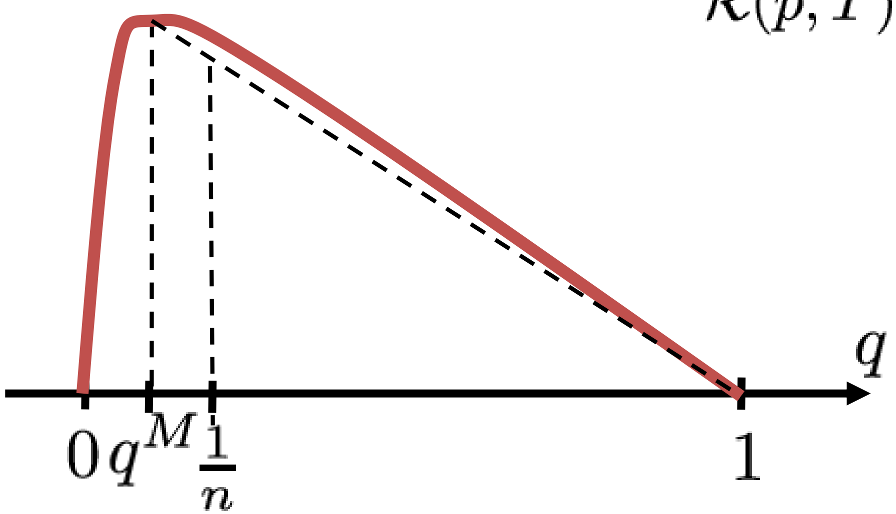
Global Public Goods $G = K_n$

Now, consider p as above and the corresponding symmetric equilibrium, $T_i = F^{-1}(1 - 1/n)$

Case 2. $T \leq p^M$

$$T(1 - F(T)) \geq (1 - 1/n)p^M(1 - F(p^M))$$

$$\begin{aligned}\mathcal{R}(p,T) &= nT(1 - F(T))F(T)^{n-1} \geq \\ &\geq (1 - 1/n)np^M(1 - F(p^M)) \\ &\geq (1 - 1/n)^n\mathcal{R}^M \geq \frac{1}{4}\mathcal{R}^M\end{aligned}$$



Global Public Goods $G = K_n$

Now, consider p as above and the corresponding symmetric equilibrium, $T_i = F^{-1}(1 - 1/n)$

Omitted here:
For this uniform price p , the revenue of any equilibrium is at least $\frac{1}{2}$ of the revenue of the symmetric equilibrium.