On Revenue in the Generalized Second Price Auction *

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ABSTRACT

The Generalized Second Price (GSP) auction is the primary auction used for selling sponsored search advertisements. In this paper we consider the revenue of this auction at equilibrium. We prove that if agent values are drawn from identical regular distributions, then the GSP auction paired with an appropriate reserve price generates a constant fraction (1/6th) of the optimal revenue.

In the full-information game, we show that at any Nash equilibrium of the GSP auction obtains at least half of the revenue of the VCG mechanism excluding the payment of a single participant. This bound holds also with any reserve price, and is tight.

Finally, we consider the tradeoff between maximizing revenue and social welfare. We introduce a natural convexity assumption on the click-through rates and show that it implies that the revenue-maximizing equilibrium of GSP in the full information model will necessarily be envy-free. In particular, it is always possible to maximize revenue and social welfare simultaneously when click-through rates are convex. Without this convexity assumption, however, we demonstrate that revenue may be maximized at a non-envy-free equilibrium that generates a socially inefficient allocation.

Categories and Subject Descriptors

J.4 [**Social and Behavioral Sciences**]: Economics; F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems

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1. INTRODUCTION

The sale of sponsored search advertising space is a primary source of income for Internet companies, and responsible for billions of dollars in annual advertising revenue [7]. The Generalized Second Price (GSP) auction is the premier method by which sponsored search advertising space is sold; it is currently employed by Google, Bing, and Yahoo. However, use of the GSP auction is not universal: the classical VCG mechanism was adopted by Facebook for its AdAuction system [12]. In fact, Google also considered switching its advertising platform to a VCG auction some years ago, but eventually decided against it [23]. This apparent tension underlines the importance of understanding how these mechanisms compare. There are many factors in comparing possible mechanisms: The welfare of three distinct user groups (the experience of the searchers, the welfare of advertisers, and the revenue of the auction) are all important considerations, as well as the simplicity of the auction design. In this paper, we take the point of view of the seller and compare the revenue properties of the GSP and VCG auctions.

Previous studies of the revenue of GSP have largely focused on outcomes of the full information game, restricted to the subclass of envy-free equilibria [7, 25, 9]. Here we consider revenue in equilibria of the Bayesian partial information version of the game, as well as equilibria of the full information game that are not envy-free, and are interested to show that GSP generates close to as much revenue as the classical optimal VCG auction. This comparison is natural, as in the Bayesian model the VCG auction is revenue optimal with the appropriate reserve price [20].

Let us first briefly describe the model introduced by Edelman et al [7] and Varian [24]. In sponsored search, a user makes a query for certain keywords in a search engine and is presented with a list of relevant advertisements in addition to organic search results. We assume a "pay-per-click" pricing model, in which the advertiser pays a fee to the search provider whenever a user clicks on an advertisement. There are multiple positions (or "slots") in which an ad may appear, and the probability that a user clicks an ad depends on its slot. This is modeled as a click-through rate (CTR), a probability α associated with each slot, the probability of getting a click for an advertisement in that position. The search engine must therefore determine which ads to place where, and determine a price per click for each slot. This is done via an auction in which advertisers make bids, which are viewed as the advertiser's maximum willingness to pay per click. We note that this simplified model as a one-shot game does not account for advertiser budgets, so models the case when budgets are large. Also, for simplicity of presentation, we will assume that all ads have the same quality (i.e. click-through rate does not depend on the advertiser selected for a slot), though our results for the full information game extend to the version of the model with separable click-though rates.

The VCG and GSP mechanisms differ in the way in which the aforementioned auction is resolved. In both auctions, advertisers are assigned slots in order of their bids, with higher bidders receiving slots with higher click probabilities. The two auctions differ in their payment schemes. In VCG, each agent pays an amount equal to his externality on the other agents: the decrease in the total welfare of all other agents caused by the presence of this advertiser. By contrast, in GSP each advertiser simply pays a price per click equal to the next highest bid. The VCG auction has the strong property of being truthful in dominant strategies. The GSP auction is not truthful, and is therefore prone to strategic bidding behavior. Indeed, strategic manipulation of bids is well-documented in historical GSP bidding data [6].

Since the VCG mechanism is truthful, the revenue of VCG is simply the revenue generated when all bidders declare their values truthfully. If bidders declare their values truthfully in a GSP auction, GSP generates strictly more revenue than VCG. However, rational agents may not declare their values truthfully when participating in a GSP auction. Thus, when studying the revenue of GSP, we consider the revenue generated at a Nash equilibrium; that is, a profile of bidding strategies such that no advertiser can improve his utility (or expected utility in the Bayesian case) by unilaterally deviating. Our goal, then, is a comparison between the revenue of the VCG auction and the revenue of GSP at equilibrium.

Results.

In Section 3 we consider the Bayesian version of this game when valuations are drawn from identical and independent distributions that satisfy the regularity condition. We show that if we allow the auctioneer to include reserve prices the GSP auction always obtains a constant fraction (1/6th) of the optimal VCG revenue, in expectation. The Bayesian setting is motivated by player uncertainty: for many keywords, the ability of a player to exactly predict his opponents types is impaired. Due to complicating factors such as the budgets, quality scores (which depend on many characteristics of each query, such as origin, time, and search history of the user), and the underlying ad allocation algorithm, each auction is different (even those triggered by the same search term) [22]. The resulting uncertainty is captured by equilibria in a Bayesian partial information model, where bidders can only evaluate the expected welfare of the equilibrium, with expectation taken over the valuations of other bidders. It is well-known that, in this setting the revenue-optimal truthful mechanism is the VCG auction with Myerson's reserve price [20].

One might also wish to bound the revenue of GSP with respect to the revenue of VCG without reserve prices, but we show that this is not possible: there are cases in which the VCG revenue is unboundedly greater than the GSP revenue. However, if the slot CTRs satisfy a certain well-separatedness condition - namely that the click through rates of adjacent slots differ by at least a certain constant factor - then we prove that GSP always obtains a constant fraction of the VCG revenue even in settings of partial information, extending a result of Lahaie [15] who considered welfare under this assumption on the CTRs. Our result holds even if we do not assume that agents avoid dominated strategies, as long as there are at least three participants in the auction.

In Sections 4 and 5 we consider the full information game. When auctions with the same participants, valuations, and quality scores are repeated many times each day, a stable outcome can be modeled by a full-information Nash equilibrium of the one-shot game. We prove that at any Nash equilibrium, the revenue generated by GSP is at least half of the VCG revenue, *excluding the single largest payment of a bidder*. Thus, as long as the VCG revenue is not concentrated on the payment of a single participant, the worst-case GSP revenue approximates the VCG revenue to within a constant factor. This result also holds with an arbitrary reserve price. We also provide an example illustrating that the factor of 2 in our analysis is tight, and the revenue of GSP at equilibrium may be arbitrarily less than the full revenue of VCG (without excluding a bidder).

In Section 5 we analyze the tradeoffs of the maximum revenue attainable by the full information GSP mechanism under different equilibrium notions. We demonstrate that there can exist inefficient, non-envy-free equilibria that obtain greater revenue than any envy-free equilibrium. However, we prove that if CTRs are *convex*, meaning that the marginal increase in CTR is monotone in slot position, then the optimal revenue always occurs at an envy-free equilibrium. This implies that when click-through rates are convex, the GSP auction optimizes revenue at an equilibrium that simultaneously maximizes the social welfare. The convexity assumption we introduce is quite natural and may be of independent interest. Note that this assumption is satisfied in the case when CTRs degrade by a constant factor from one slot to the next.

Related Work.

There has been considerable amount of work on the economic and algorithmic issues behind sponsored search auctions – see the survey of Lahaie et al [16] for an overview of the early work and the survey of Maille et al [18] for recent developments. The GSP model we adopt is due to Edelman et al [7], Varian [24] and Aggarwal et al [2]. Much of the previous work on the GSP auction considered social welfare properties of equilibria. Edelman et al [7] and Varian [24] define a subclass of Nash equilibria called envy-free equilibria and show that such equilibria always exist and are socially optimal, therefore showing that the price of stability is 1 for the full information game. Paes Leme and Tardos [21] showed a bound of 1.618 on the price of anarchy, which was recently improved to 1.282 by Caragiannis at al [4]. The Bayesian version of this game, when valuations are random and only the distribution is public knowledge, was first studied by Gomes and Sweeney [10] who showed that efficient equilibria may not exist in this setting. Paes Leme and Tardos [21] prove a price of anarchy of 8, which was recently improved by Lucier and Paes Leme [17] and Caragiannis at al [4] to 3.03.

Considering revenue properties of GSP, Edelman et al [7], and Varian [24] show that envy free equilibria have revenue at least as good as the revenue of VCG. Both [7] and [24] present informal arguments to support the equilibrium selection for this class of equilibria, but there is no strong theoretical model that explains this selection [3, 9]. Further, the notion of envy-free equilibria applies only in the full information game. Varian [25] shows how to compute the revenue optimal envy-free Nash equilibrium, although his model allows agents to overbid (which is dominated strategy, and we consider it unnatural). We consider the question of maximum revenue equilibria without the assumption of envy-free outcomes, and show that in general inefficient equilibria can generate more revenue than efficient ones, but this is no longer the case under a natural convexity assumption.

Gomes and Sweeney [10] study GSP as a Bayesian game, and show that symmetric efficient equilibria may not exist in the Bayesian setting. They analyze the influence of click-through-rates on the revenue in equilibrium and observe the counter-intuitive phenomenon by which revenue decreases when click-through-rates increase.

A common tool for increasing revenue in settings of partial information is to apply *reserve prices*, where bids are rejected unless they meet some minimum bid r. When bidders' values are drawn from identically distributed satisfying the regularity condition[1], the revenue-optimal truthful auction for sponsored search is the VCG auction with an appropriate reserve price [20]. Edelman and Schwarz [8] show that in GSP auctions reserve prices have a surprisingly large effect on revenue.

Edelman and Schwarz [9] model the repeated auctions for a keyword as a repeated game, and show, using Myerson's optimal auction [20], that if valuations are drawn from an iid distribution then the Nash equilibria that arise as a stable limit of rational play in this repeated game cannot have revenue more than the optimal auction: VCG with an appropriately chosen reserve price. However, they don't consider whether GSP may generate revenue much less than the VCG auction, which is the main question we consider. Also, unlike Edelman and Schwarz [9], we do not use the repeated nature of this game for arguing for certain equilibrium selection: rather, we consider *all* stable outcomes of the auction, not only those that arise as limits of rational repeated play.

Finally, we mention that the well-known *revenue equivalence theorem*, which provides conditions under which alternative mechanisms generate the same revenue at equilibrium, does not apply in our settings. Revenue equivalence requires that agents have values drawn from identical distributions and that the mechanisms generate the same outcome. As a result, this equivalence does not apply in the full information setting, and cannot be used to compare inefficient equilibria of GSP to the VCG revenue.

2. PRELIMINARIES

An AdAuctions instance is composed of n players and n slots. In the full information model, each player has a value v_i for each click he gets, and each slot j has click-through-rate α_j. That means that if player i is allocated in slot j, he gets α_j clicks in expectation, where $\alpha_1 \geq \ldots \geq \alpha_n$. In the full information setting, we will assume players are numbered such that $v_1 \geq v_2 \geq \ldots \geq v_n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be the CTR vector and $\mathbf{v} = (v_1, \ldots, v_n)$ be the type vector.

[1]Many common distributions are regular, including all uniform, normal, and exponential distributions.

A mechanism for the AdAuctions problem is as follows: it begins by eliciting bids b_i from the players, which can be thought of as declared valuations. We write $\mathbf{b} = (b_1, \ldots, b_n)$ for the bid vector. Using the $\mathbf{b}$ and α, the mechanism chooses an allocation $\pi : [n] \to [n]$ which means that player $\pi(j)$ is allocated to slot j, and a price vector $\mathbf{p} = (p_1, \ldots, p_n)$, where p_i is the price that player i pays per click. Player i then experiences utility $u_i(\mathbf{b}) = \alpha_{\sigma(i)}(v_i - p_i)$, where $\sigma(i) = \pi^{-1}(i)$.

The social welfare generated by the mechanism is given by $SW(\mathbf{v}, \pi) = \sum_i \alpha_i v_{\pi(i)}$ and the revenue is given by $\mathcal{R}(\mathbf{b}) = \sum_i \alpha_{\sigma(i)} p_i$. We focus on two mechanisms: GSP and VCG. In both mechanisms, the players are ordered by their bids, i.e, $\pi(j)$ is the player with the j^{th} largest bid, but they differ in the payments charged. GSP mimics the single-item second price auction by charging each player the bid of the next highest bidder, i.e. $p_i = b_{\pi(\sigma(i)+1)}$ if $\sigma(i) < n$ and zero otherwise. VCG charges each player the externality it imposes on the other players, which is $p_i^{VCG} = \frac{1}{\alpha_{\sigma(i)}} \sum_{j=\sigma(i)+1}^{n} (\alpha_{j-1} - \alpha_j) b_{\pi(j)}$.

VCG is a truthful mechanism: regardless of what the other players are doing, it is a weakly dominant strategy for player i to report his true valuation. The resulting outcome of VCG is therefore social-welfare optimal and the revenue is

$$\mathcal{R}^{VCG}(\mathbf{v}) = \sum_i \sum_{j>i} (\alpha_{j-1} - \alpha_j) v_j = \sum_{i=2}^{n} (i-1)(\alpha_{i-1} - \alpha_i) v_i.$$

The GSP auction, however, is not truthful. Thus, for GSP, we are interested in the set of bid profiles that constitute a Nash equilibrium, i.e. such that

$$u_i(b_i, \mathbf{b}_{-i}) \geq u_i(b_i', \mathbf{b}_{-i}), \forall b_i' \in [0, v_i].$$

We will assume that players do not overbid (i.e. that $b_i \leq v_i$) since bidding more than one's true value is a weakly dominated strategy [21].

We say that an equilibrium is **efficient** if it maximizes social welfare, i.e. $\pi(i) = i, \forall i$ in the full information version.

We will also consider the comparison between VCG and GSP in the presence of a reserve price. Let VCG_r be the VCG mechanism with reserve price r, where we discard all players with bids smaller then r and run the VCG mechanism on the remaining players, who then pay price per click $\max\{p_i, r\}$. In the analogous variant of the GSP mechanism, which we call GSP with reserve price r (GSP_r), we also discard all players with bids smaller then r, the remaining players are allocated using GSP, and the last player to be allocated pays price r per click.

Below, we represent the special classes of equilibria that have been studied in the literature, which we call **equilibrium hierarchy** for GSP. We define and discuss them in Section 5 :

$$\left\{ \begin{array}{c} \text{VCG} \\ \text{outcome} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{envy-free} \\ \text{equilibria} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{efficient} \\ \text{Nash eq} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{all} \\ \text{Nash} \end{array} \right\}$$

2.1 Bayesian setting

In a Bayesian setting, each player knows her own valuation but only knows a distribution on the other players' valuations. In this model the values of the players are random variables, with type vector $\mathbf{v}$ drawn from a known distribution F. After learning his own value v_i, a player chooses a bid $b_i(v_i)$ to play in the AdAuctions game. The strategies are therefore bidding functions $b_i : \mathbb{R}_+ \to \mathbb{R}_+$, and we will continue to assume that players do not overbid, i.e. $b_i(v) \le v$ (again, since overbidding is weakly dominated). A set of bidding functions is a **Bayes-Nash equilibrium** if, for all i, v_i, and b'_i,

$$\mathbb{E}[u_i(b_i(v_i), \mathbf{b}_{-i}(\mathbf{v}_{-i}))|v_i] \ge \mathbb{E}[u_i(b'_i(v_i), \mathbf{b}_{-i}(\mathbf{v}_{-i}))|v_i].$$

A useful tool for studying revenue in the Bayesian setting is Myerson's Lemma, which can be rephrased in the AdAuctions setting as follows. Given a distribution F over agent values, the **virtual valuation** function is defined by $\phi(x) = x - \frac{1-F(x)}{f(x)}$.

Lemma 1 (Myerson [20]) *At any Bayes-Nash equilibrium of an AdAuction mechanism, we have that, for all i, $\mathbb{E}[\alpha_{\sigma(i)} p_i] = \mathbb{E}[\alpha_{\sigma(i)}\phi(v_i)]$ where p_i is the payment per click of player i and $\alpha_{\sigma(i)}$ is the number of clicks received by agent i, and expectation is with respect to $\mathbf{v} \sim \mathbf{F}$.*

We say that a distribution is **regular** if $\phi(x)$ is a monotone non-decreasing function. For regular distributions, the revenue-optimal mechanism for AdAuctions corresponds to running VCG with Myerson's reserve price r, which is the largest value such that $\phi(r) = 0$. We will refer to this as *Myerson's mechanism*, VCG_r.

Running GSP (or VCG) with **reserve price** r means not allocating any user with value $v_i < r$ and running GSP (or VCG) with the remaining agents. For the allocated agents, the mechanism charges per click the maximum between the GSP (VCG) price and r.

A special class of regular distributions is the **monotone hazard rate** distributions (**MHR**), which are the distributions for which $f(x)/(1 - F(x))$ is non-decreasing.

3. REVENUE IN THE BAYESIAN SETTING

In this section we consider the revenue properties of GSP at Bayes-Nash equilibrium. We prove that if agent values are drawn iid from a regular distribution and GSP is paired with an appropriate reserve price, the revenue generated at equilibrium will be within a constant factor of the VCG revenue with optimal reserve, the revenue-optimal mechanism over all Bayes-Nash implementations. So our result implies that GSP revenue is within a constant factor of the optimal. We will first consider the special case of MHR distributions, then prove our result in the more general setting where values are drawn from regular distributions.

We start by showing that the use of reserve prices is crucial: there are instances in which the GSP auction without reserve generates no revenue, whereas the VCG auction generates positive revenue. We do note, however, that one can bound the revenue of GSP without reserve prices when click-through rates are *well-separated*, meaning that there exists a δ such that $\alpha_{i+1} \le \delta\alpha_i$ for all i. We present these bounds in the Appendix A..

3.1 Revenue without Reserves: Bad Examples

We start by providing an example in the Bayesian setting where VCG generates positive revenue and GSP has a Bayes-Nash equilibrium that generates zero revenue. Consider three players with iid valuations drawn uniformly from $[1, 2]$ and three slots with $\alpha = [1, 0.5, 0.5]$. Let $v^{(i)}$ be the i^{th} largest valuation (which is naturally a random variable defined by $\mathbf{v}$). We have

$$\mathbb{E}[\mathcal{R}^{VCG}(\mathbf{v})] = \mathbb{E}[0.5v^{(2)}] = \frac{3}{4}.$$

Now, consider the following equilibrium of GSP: $b_i(v_i) = 0$ for $i = 2, 3$ and $b_1(v_1) = v_1$. Clearly player 1 is in equilibrium. To see that players $i = 2, 3$ are in equilibrium, suppose player i has valuation $v_i > 0$. Then his expected utility when bidding any value in $[0, 1]$ is $0.5v_i$, whereas if he changed his bid to some $b > 1$ his utility would be

$$\mathbb{E}[u_i(b', b_{-i})|v_i] = 0.5v_i + 0.5v_i\mathbb{P}(v_1 \le b') - \int_0^{b'} v_1 d\mathbb{P}(v_1) =$$
$$= 0.5v_i + 0.5v_i(b' - 1) - \frac{(b')^2 - 1}{2} \le$$
$$\le 0.5v_i.$$

Thus agent i cannot increase his expected utility by placing a non-zero bid.

3.2 Warmup: MHR Valuations

We now show that if valuations are drawn from a MHR distribution and GSP is paired with the Myerson reserve price, the resulting mechanism extracts a constant fraction of the optimal revenue.

In what follows we will write x^+ to denote $\max\{x, 0\}$.

Theorem 2 *If valuations are drawn iid from a MHR distribution F and r is the Myerson reserve price for F, then the expected revenue of GSP_r at any Bayes-Nash equilibrium is at least $\frac{1}{6}$ of the optimal revenue.*

Our proof will make use of the fact that, for MHR distributions, $\phi(x) \ge x - r$ for any $x \ge r$. To see this, note that $x - \phi(x) = \frac{1-F(x)}{f(x)} \le \frac{1-F(r)}{f(r)} = r$ by monotonicity and the definition of Myerson's reserve price.

Proof. Let $\mathbf{b}$ be a Bayes-Nash equilibrium of GSP$_r$, and let $\mathcal{R}_r(\mathbf{v})$ be the expected revenue of GSP$_r$ at this equilibrium. Let $\mathcal{R}_r^{VCG}(\mathbf{v})$ be the VCG$_r$ revenue. Let random variable $\mu(i)$ denote the slot occupied by player i in the optimal (i.e. efficient) allocation. By Myerson's Lemma, $\mathbb{E}[\mathcal{R}_r^{VCG}(\mathbf{v})] = \mathbb{E}[\sum_i \alpha_{\mu(i)}\phi(v_i)^+]$. For each player i, let E_1^i denote the event that $b_{\pi(\mu(i))} < v_i/2$, and let E_2^i denote the event that $b_{\pi(\mu(i))} \ge v_i/2$. We will consider each of these events in turn. For the first event, we'll show that player i contributes to the revenue at least $1/2$ his contribution in the optimum. Consider a player i with value v_i. We have

$$\mathbb{E}_{v_{-i}}\left[\alpha_{\mu(i)}\frac{v_i}{2}\mathbb{1}\{E_1^i\}\right] \le \mathbb{E}_{v_{-i}}\left[u_i\left(\frac{v_i}{2}, \mathbf{b}_{-i}\right)\right]$$
$$\le \mathbb{E}_{v_{-i}}[u_i(\mathbf{b})] \le \mathbb{E}_{v_{-i}}[\alpha_{\sigma(i)}v_i]$$

where the first inequality is due to the definition of E_1^i implying that a bid of $v_i/2$ would win slot $\mu(i)$ (or better) at price no more than $v_i/2$; the second follows since $\mathbf{b}$ is a Bayes-Nash equilibrium, and the third comes from the definition of utility. Notice that all the expectations are taken over v_{-i} and v_i is a constant, so we can divide by v_i, multiply by $\phi(v_i)^+$, take expectations over v_i and sum over all players i to obtain

$$\sum_i \mathbb{E}_{\mathbf{v}}[\alpha_{\mu(i)}\phi(v_i)^+\mathbb{1}\{E_1^i\}] \le 2\sum_i \mathbb{E}_{\mathbf{v}}[\alpha_{\sigma(i)}\phi(v_i)^+]$$
$$= 2\mathbb{E}_{\mathbf{v}}[\mathcal{R}_r(\mathbf{v})].$$

For the second event, consider again a player i with value v_i. We will show that the player who gets slot $\mu(i)$ contributes to the revenue. We have

$$\begin{aligned}\mathbb{E}_{v_{-i}}\left[\alpha_{\mu(i)}\frac{\phi(v_i)^+}{2}\mathbb{1}\{E_2^i\}\right] &\leq \mathbb{E}_{v_{-i}}\left[\alpha_{\mu(i)}\frac{v_i}{2}\mathbb{1}\{E_2^i\}\right] \\ &\leq \mathbb{E}_{v_{-i}}[\alpha_{\mu(i)}v_{\pi(\mu(i))}] \\ &\leq \mathbb{E}_{v_{-i}}[\alpha_{\mu(i)}(r+\phi(v_{\pi(\mu(i))})^+)]\end{aligned}$$

where we used the fact that $x \geq \phi(x)^+ \geq x - r$ for all x. Taking expectations over v_i, summing over all players, and noting that event E_2^i implies that $v_{\pi(\mu(i))} \geq r$, we obtain

$$\begin{aligned}&\sum_i \mathbb{E}_{\mathbf{v}}[\alpha_{\mu(i)}\phi(v_i)^+\mathbb{1}\{E_2^i\}] \leq \\ &\leq 2\sum_i \mathbb{E}_{\mathbf{v}}[\alpha_{\sigma(i)}\phi(v_i)^+] + 2\sum_i \alpha_{\sigma(i)} r \mathbb{1}\{v_i \geq r\}.\end{aligned}$$

Since GSP_r extracts a revenue of at least r per click from every bidder with $v_i > r$, we have $\mathbb{E}_{\mathbf{v}}[\mathcal{R}_r] \geq \mathbb{E}_{\mathbf{v}}[\sum_i \alpha_{\sigma(i)} r \mathbb{1}\{v_i \geq r\}]$. We conclude that $\sum_i \mathbb{E}_{v_{-i}}[\alpha_{\mu(i)}\phi(v_i)^+\mathbb{1}\{E_2^i\}] \leq 4\mathbb{E}[\mathcal{R}_r]$. Combining our analysis for the two events, we have

$$\begin{aligned}\mathbb{E}[\mathcal{R}_r^{VCG}(\mathbf{v})] &= \mathbb{E}\left[\sum_i \alpha_{\mu(i)}\phi(v_i)^+(\mathbb{1}\{E_1^i\} + \mathbb{1}\{E_2^i\})\right] \\ &\leq 2\mathbb{E}[\mathcal{R}_r(\mathbf{v})] + 4\mathbb{E}[\mathcal{R}_r(\mathbf{v})] = 6\mathbb{E}[\mathcal{R}_r(\mathbf{v})].\end{aligned}$$

■

3.3 Regular valuations

We now show that if player valuations are drawn from a regular distribution, then there exists an r' such that running GSP with reserve r' extracts a constant fraction of the optimal revenue. The bound for the MHR bounding the contribution of the player at slot $\mu(i)$ took advantage of the fact that in a MHR distribution $\phi(x) \geq x - r$, which may not be true in a regular distribution. Instead, we will use that the player in slot $\mu(i) - 1$ pays at least the bid in slot $\mu(i)$. This leaves us with the added difficulty in bounding the revenue generated by the first slot. To address this issue, we make use of the well-studied Prophet Inequalities [13, 14, 11].

A simplified version of the Prophet Inequality is as follows. Suppose z_i are independent non-negative random variables. Given any $t \geq 0$, write y_t for the value of the first z_i (by index) satisfying $z_i > t$ (or 0 if there is no such z_i). Then the Prophet Inequality states that there exists some $t \geq 0$ such that $\mathbb{E}[y_t] \geq \frac{1}{2}\mathbb{E}[\max_i z_i]$. Since the proof is of this fact is very short, we include it for completness in Appendix B.

As has been noted elsewhere [5], the Prophet Inequality has immediate consequences for the revenue of auctions with anonymous reserve prices. The following lemma encapsulates the observation we require.

Lemma 3 *If v_i are drawn iid from a regular distribution then there exists $r_2 \geq 0$ such that, writing Z for the event that $\max_i v_i \geq r_2$, $\mathbb{E}[\max_i \phi(v_i)^+|Z]\mathbb{P}(Z) \geq \frac{1}{2}\mathbb{E}[\max_i \phi(v_i)^+]$.*

PROOF. (sketch) This follows by applying the Prophet Inequality to virtual values $z_i = \phi(v_i)$ and noting that regularity implies that $v_i \geq r_2$ iff $\phi(v_i) \geq \phi(r_2)$. ■

It is important to remark that the proof of the Prophet Inequality is constructive. If one is able to efficiently compute $\mathbb{E}[(v_i - t)^+]$ for every t, then we can compute r_2 exactly using binary search. We refer to Appendix B for details.

Our approach will now be to analyze the revenue of GSP under two different reserve prices. An argument similar to Theorem 2 shows that GSP with Myerson reserve obtains a constant fraction of the optimal revenue for all slots other than the first slot. On the other hand, GSP with reserve r_2 from Lemma 3 will obtain at least half of the optimal revenue generated by the first slot. One of these two reserve prices must therefore generate a constant fraction of the optimal revenue.

Theorem 4 *If valuations v_i are drawn iid from a regular distribution F, then there is a reserve price r such that the expected revenue of GSP_r at any Bayes-Nash equilibrium is at least $\frac{1}{6}$ of the optimal revenue.*

PROOF. Define $\mathcal{R}_r^{VCG}(\mathbf{v})$, $\mathcal{R}_r(\mathbf{v})$, and $\mu(i)$ as in Theorem 2. Let r_1 denote the Myerson reserve price for F. By Myerson's Lemma, $\mathbb{E}[\mathcal{R}_r^{VCG}(\mathbf{v})] = \mathbb{E}[\sum_i \alpha_{\mu(i)}\phi(v_i)^+]$. For each player i, we define the following three events:

- $E_1^i = \{\ b_{\pi(\mu(i))} < v_i/2 \text{ and } \mu(i) \neq 1\ \}$
- $E_2^i = \{\ b_{\pi(\mu(i))} \geq v_i/2 \text{ and } \mu(i) \neq 1\ \}$
- $E_3^i = \{\ \mu(i) = 1\ \}$

We wish to bound the virtual value of the optimal allocation, conditioning on each of these events in turn. For the first event, we proceed precisely as in Theorem 2 to obtain

$$\sum_i \mathbb{E}_{v_{-i}}[\alpha_{\mu(i)}\phi(v_i)^+\mathbb{1}\{E_1^i\}] \leq 2\mathbb{E}[\mathcal{R}_{r_1}].$$

For the second event, we use the revenue from slot $\mu(1) - 1$. Let random variable p_i denote the payment per click of the player in slot i. Then for all $\mathbf{v}$,

$$\alpha_{\mu(i)}\phi(v_i)^+\mathbb{1}\{E_2^i\} \leq \alpha_{\mu(i)}v_i\mathbb{1}\{E_2^i\} \leq 2\alpha_{\mu(i)-1}p_{\mu(i)-1}\mathbb{1}\{E_2^i\}$$

where the second inequality follows since E_2^i implies $p_{\mu(i)-1} = b_{\mu(i)} \geq v_i/2$. Therefore, summing over all agents i and taking expectations, we get

$$\mathbb{E}_{\mathbf{v}}\left[\sum_i \alpha_{\mu(i)}\phi(v_i)^+\mathbb{1}\{E_2^i\}\right] \leq 2\mathbb{E}_{\mathbf{v}}\left[\sum_i \alpha_i p_i\right] = 2\mathbb{E}[\mathcal{R}_{r_1}].$$

Finally, for event E_3^i, consider setting the reserve price to be r_2 from the statement of Lemma 3 (with distribution F). Note that

$$\mathbb{E}\left[\sum_i \alpha_{\mu(i)}\phi(v_i)^+\mathbb{1}\{E_3^i\}\right] = \alpha_1\mathbb{E}[\max_i \phi(v_i)^+].$$

On the other hand, setting reserve price r_2 for GSP we get

$$\begin{aligned}\mathbb{E}[\mathcal{R}_{r_2}] &\geq \alpha_1\mathbb{E}[\max_i \phi(v_i)^+ \mid \max_i v_i \geq r_2]\mathbb{P}(\max_i v_i \geq r_2) \\ &\geq \frac{1}{2}\alpha_1\mathbb{E}[\max_i \phi(v_i)^+]\end{aligned}$$

where the first inequality follows by considering only the expected virtual value due to the first slot and the last inequality follows from Lemma 3. Combining our analysis for each of the three cases, we have

$$\begin{aligned}\mathbb{E}[\mathcal{R}_r^{VCG}] &= \mathbb{E}\left[\sum_i \alpha_{\mu(i)}\phi(v_i)^+(\mathbb{1}\{E_1^i\} + \mathbb{1}\{E_2^i\} + \mathbb{1}\{E_3^i\})\right] \\ &\leq 4\mathbb{E}[\mathcal{R}_{r_1}] + 2\mathbb{E}[\mathcal{R}_{r_2}]\end{aligned}$$

and hence $\max\{\mathbb{E}[\mathcal{R}_{r_1}], \mathbb{E}[\mathcal{R}_{r_2}]\} \geq \frac{1}{6}\mathbb{E}[\mathcal{R}_r^{VCG}]$. ■

4. REVENUE IN FULL INFORMATION GSP

We now wish to compare the revenue properties of GSP and VCG in the full information setting. We start by giving examples showing that there are no universal constants that bound these two quantities. Then we introduce a new benchmark related to VCG, and show that the GSP revenue is not too low relative to this benchmark.

4.1 Full Information Revenue: Examples

Unfortunately, there are no universal constants $c_1, c_2 > 0$ such that for every full information AdAuctions instance $\alpha, \mathbf{v}$ and for all equilibria b of GSP it holds that

$$c_1 \cdot \mathcal{R}^{VCG}(\mathbf{v}) \leq \mathcal{R}(\mathbf{b}) \leq c_2 \cdot \mathcal{R}^{VCG}(\mathbf{v}).$$

In fact, GSP can generate arbitrarily more revenue than VCG and vice-versa. For example, consider two players with $\alpha = \{1, 0\}$, $\mathbf{v} = \{2, 1\}$. Then VCG generates revenue 1, but GSP has the Nash equilibrium $\mathbf{b} = [2, 0]$ that generates no revenue.

As a counter-example for the second inequality, consider the following instance: $\alpha = \{1, 1-\epsilon\}$, $\mathbf{v} = \{\epsilon^{-1}, 1\}$. Notice that the revenue produced by VCG is ϵ, while GSP has the equilibrium $\mathbf{b} = [1, 1]$ generating revenue 1.

4.2 Revenue Bound in Full Information

Next, we will prove that the GSP revenue cannot be much less than a revenue benchmark based on the VCG auction. Intuitively, the difficulty behind our bad examples is in extracting revenue from the player with the largest private value. Motivated by this, we consider the following benchmark:

$$\mathcal{B}(\mathbf{v}) = \sum_{i=2}^{n} p_i^{VCG} \alpha_{\sigma(i)} = \sum_{i=2}^{n} \sum_{j>i} (\alpha_{j-1} - \alpha_j) v_j$$
$$= \sum_{i=2}^{n} (i-2)(\alpha_{i-1} - \alpha_i) v_i$$

which is the VCG revenue from players $2, 3, \ldots, n$. Recall that in the full information setting we assumed that players are numbered such that $v_1 \geq v_2 \geq \ldots$. We show that the GSP revenue is always at least half of this benchmark at any equilibrium. Thus, unless VCG gets most of its revenue from a single player, GSP revenue will be within a constant factor of the VCG revenue.

Theorem 5 *Given an AdAuctions instance $\alpha, \mathbf{v}$, and a Nash equilibrium $\mathbf{b}$ of GSP, we have $\mathcal{R}(\mathbf{b}) \geq \frac{1}{2}\mathcal{B}(\mathbf{v})$, and this bound is tight.*

We prove Theorem 5 in two steps. First we define the concept of up-Nash[2] equilibrium for GSP, then we show that any inefficient Nash equilibria can be written as an efficient up-Nash equilibrium. In the second step, we prove the desired revenue bound for all efficient up-Nash equilibria.

Definition 6 *Given a bid profile $\mathbf{b}$, we say it is **up-Nash** for player i if he can't increase his utility by taking some slot above, i.e.*

$$\alpha_{\sigma(i)}(v_i - b_{\pi(\sigma(i)+1)}) \geq \alpha_j (v_i - b_{\pi(j)}), \forall j < \sigma(i).$$

*Analogously, we say that $\mathbf{b}$ is **down-Nash** for player i if he can't increase his utility by taking some slot below, i.e.*

$$\alpha_{\sigma(i)}(v_i - b_{\pi(\sigma(i)+1)}) \geq \alpha_j (v_i - b_{\pi(j+1)}), \forall j > \sigma(i).$$

A bid profile is up-Nash (down-Nash) if it is up-Nash (down-Nash) for all players i. Clearly a bid profile $\mathbf{b}$ is a Nash equilibrium iff it is both up-Nash and down-Nash.

Lemma 7 *If a bid profile $\mathbf{b}$ is a Nash equilibrium, then the bid profile $\mathbf{b}'$ where $b'_i = b_{\pi(i)}$ is up-Nash.*

Proof. We will prove the lemma by modifying bid profile $\mathbf{b}$ in a sequence of steps. Fix some $k \leq n$, and suppose that $\mathbf{b}$ is a bid profile (with corresponding allocation π) such that

- players $j = 1, \ldots, k$ satisfy the Nash conditions (i.e. both up-Nash and down-Nash) in $\mathbf{b}$,
- players $j = k+1, \ldots, n$ are such that $\sigma(j) = j$ and they satisfy the up-Nash conditions in $\mathbf{b}$,
- $\sigma(k) < k$.

We then define $\mathbf{b}'$ by swapping the bids of players k and $\pi(k)$, that is setting $b'_i = b_i$ for $i \neq k, \pi(k)$, $b'_k = b_{\pi(k)}$, and $b'_{\pi(k)} = b_k$. We claim that $\mathbf{b}'$ is up-Nash for players $k, \ldots, n$ and Nash for the remaining players. This then implies the desired result, since we can apply this operation for $k = n$, followed by $k = n-1, \ldots, 2$, resulting in the required bid profile.

Since our transformation does not alter the bids associated with given slots, we just need to check three things: the up and down-Nash inequalities for player $\pi(k)$, and the up-Nash inequality for player k.

Under bid profile $\mathbf{b}'$, player $\pi(k)$ gets slot $\sigma(k)$. This player doesn't want to change his bid to win any slot $j > \sigma(k)$ since in the bid profile $\mathbf{b}$ player k with lower value didn't want to get these slots. We therefore have $\alpha_{\sigma(k)}(v_k - b_{\pi(\sigma(k)+1)}) \geq \alpha_j(v_k - b_{\pi(j+1)})$ and since $v_{\pi(k)} \geq v_k$, we conclude

$$\alpha_{\sigma(k)}(v_{\pi(k)} - b_{\pi(\sigma(k)+1)}) \geq \alpha_j(v_{\pi(k)} - b_{\pi(j+1)}). \quad (1)$$

To see that player $\pi(k)$ would not prefer to take any slot $j < \sigma(k)$, notice that $\pi(k)$ didn't want to move to a higher slot in b, so $\alpha_k(v_{\pi(k)} - b_{\pi(k+1)}) \geq \alpha_j(v_{\pi(k)} - b_{\pi(j)})$. This, combined with equation (1) for $j = k$ (stating that $\pi(k)$ prefers slot $\sigma(k)$ to k) gives the up-Nash inequality for player $\pi(k)$.

Next consider player k in bid profile $\mathbf{b}'$, where we gets slot k. We wish to prove the up-Nash inequality for k. Notice that, in $\mathbf{b}$, $\pi(k)$ had slot k and didn't want to switch to a higher slot, so we know $\alpha_k(v_{\pi(k)} - b_{\pi(k+1)}) \geq \alpha_j(v_{\pi(k)} - b_{\pi(j)})$.

Now, since $v_{\pi(k)} \geq v_k$, we have $\alpha_k(v_k - b_{\pi(k+1)}) \geq \alpha_j(v_k - b_{\pi(j)})$ which is the desired inequality. ■

Now to prove Theorem 5 we use the up-Nash profile b'.

Proof of Theorem 5 : Given any Nash equilibrium $\mathbf{b}$, consider the bid profile $\mathbf{b}'$ of Lemma 7, which is an up-Nash equilibrium in which each player k occupies slot k. By the up-Nash inequalities, for each k we have

$$\alpha_k(v_k - b'_{k+1}) \geq \alpha_{k-1}(v_k - b'_{k-1}).$$

We can rewrite this as

$$\alpha_{k-1} b'_{k-1} \geq (\alpha_{k-1} - \alpha_k) v_k + \alpha_k b'_{k+1}.$$

[2]Our concepts of up-Nash and down-Nash equilibria are very similar to the concepts of upwards stable and downwards stable equilibria in Markakis and Telelis [19]

Then, since $\alpha_k \geq \alpha_{k+1}$,

$$\alpha_{k-1} b'_{k-1} \geq \sum_{j \in k+2\mathbb{N}} (\alpha_{j-1} - \alpha_j) v_j$$

where $k + 2\mathbb{N} = \{k, k+2, k+4, \ldots\}$. Now we can bound the revenue of $\mathbf{b}$:

$$\mathcal{R}(\mathbf{b}) = \mathcal{R}(\mathbf{b}') = \sum_k \alpha_k b'_{k+1} \geq \sum_k \alpha_{k+1} b'_{k+1} \geq$$
$$\geq \sum_k \sum_{j \in k+2+2\mathbb{N}} (\alpha_{j-1} - \alpha_j) v_j \geq$$
$$\geq \sum_{k=2}^{n} \frac{k-2}{2} (\alpha_{k-1} - \alpha_k) v_k = \frac{1}{2} \mathcal{B}(\mathbf{v}).$$

■

To show that the bound in Theorem 5 is tight, consider the following example with n slots and n players, parametrized by $\delta > 0$:

$$\alpha = [1, 1, \ldots, 1, 1-\delta, 0],$$
$$\mathbf{v} = [1, 1, \ldots, 1, 1, \delta],$$
$$\mathbf{b} = [\delta, \delta, \ldots, \delta, \delta, 0].$$

In this case $\mathcal{R}(\mathbf{b}) = (n-2)\delta + \delta(1-\delta)$ and $\mathcal{R}^{VCG}(\mathbf{v}) = (2\delta - \delta^2)(n-3) + \delta(1-\delta)$. Therefore $\lim_{n\to\infty} \frac{\mathcal{R}(\mathbf{b})}{\mathcal{B}(\mathbf{v})} = 2 - \delta$ and it tends to 2 as $\delta \to 0$.

Notice that those bounds also carry for the case where there is a reserve price r. We compare the revenue $\mathcal{R}_r(\mathbf{b})$ with reserve price r, against a slightly modified benchmark: $\mathcal{B}_r(\mathbf{v})$ which is the revenue VCG$_r$ extracts from players $2, \ldots, n$.

Corollary 8 *Let* $\mathbf{b}$ *be a Nash equilibrium of the GSP$_r$ game, then* $\mathcal{R}_r(\mathbf{b}) \geq \frac{1}{2}\mathcal{B}_r(\mathbf{v})$.

PROOF. We can assume wlog that $v_i, b_i \geq r$ (otherwise those players don't participate in any of the auctions). We can define an upper-Nash bid profile $\mathbf{b}'$ as in Lemma 7. Now, notice that all players in $\mathbf{b}'$ are paying at least r per click. We can divide the players in two groups: players $1 \ldots k$ are paying more than r in VCG$_r$ and players $k+1 \ldots n$ are paying exactly r. It is trivial that for the players $k+1 \ldots n$ we extract at least the same revenue under VCG$_r$ then under GSP$_r$. For the rest of the players we need to do the exact same analysis as in the proof of Theorem 5. ■

5. TRADEOFF BETWEEN REVENUE AND EFFICIENCY

In this section we consider the tradeoff between efficiency and revenue, and ask if optimal efficiency and optimal revenue can always be achieved in the same equilibrium. We give a negative answer to this question, showing that for some AdAuction instances, one can increase revenue by selecting inefficient equilibria. First we recall the equilibrium hierarchy briefly discussed in the introduction.

Then we characterize the maximum revenue possible for envy free equilibrium (that is always efficient). Does this equilibrium class generate more or less revenue than other classes, such as efficient equilibria or all pure equilibria? This question of comparing the revenue of VCG and envy-free equilibria of GSP was addressed by [7], who show that the revenue in any envy-free equilibrium is at least that of

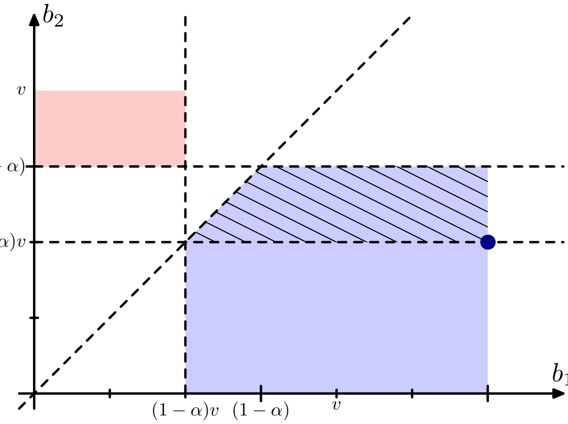

Figure 1: Equilibria hierarchy for GSP for $\alpha = [1, 1/2]$, $v = [1, 2/3]$: the strong blue dot represents the VCG outcome, the pattern region the envy-free equilibria, the blue region all the efficient equilibria and the red region the inefficient equilibria

the VCG outcome (i.e. the VCG outcome is the envy-free equilibria generating smallest possible revenue). Moreover, as we've shown, an envy-free equilibrium can generate arbitrarily more revenue than the VCG outcome. Varian [25] shows how to compute the revenue optimal envy free Nash equilibrium, if we assume that agents will overbid. Allowing overbidding can result in very high revenue (eg.., the maximum valuation in a single item auction). Here we determine the maximum revenue that can be obtained if we do not assume that agents bid at envy-free equilibria, and without requiring that agents apply the dominated strategy of overbidding.

Finally, we use this characterization to we give a natural sufficient condition under which there is a revenue-optimal equilibrium that is efficient.

5.1 Equilibrium hierarchy for GSP

Edelman, Ostrovsky and Schwarz [7] and Varian [24] showed that the full information game always has a Pure Nash equilibrium, and moreover, there is a pure Nash equilibrium which has same outcome and payments as VCG. At this equilibrium, players bid

$$b_i^V = \frac{1}{\alpha_{i-1}} \sum_{j=i}^{n} (\alpha_{j-1} - \alpha_j) v_j.$$

The authors also define a class of equilibria called **envy-free** or **symmetric equilibria**. This is the class of bid profiles $\mathbf{b}$ such that

$$\alpha_{\sigma(i)}(v_i - b_{\sigma(i)+1}) \geq \alpha_j (v_i - b_{j+1}).$$

It is easy to see that all envy-free equilibria are Nash equilibria, though not all Nash equilibria are envy-free. The bid profiles that are envy-free are always efficient and the revenue of an envy-free equilibrium is always greater than or equal to the VCG revenue. That is, if $\mathbf{b}$ is an envy-free equilibrium, $\mathcal{R}(\mathbf{b}) \geq \mathcal{R}^{VCG}(\mathbf{v})$.

Although all envy-free equilibria are efficient, there are efficient equilibria that are not envy-free, as one can see for example in Figure 1, as well as inefficient equilibria. We therefore have the following hierarchy:

$$\left\{\begin{array}{c}\text{VCG}\\ \text{outcome}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{envy-free}\\ \text{equilibria}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{efficient}\\ \text{Nash eq}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{all}\\ \text{Nash}\end{array}\right\}$$

5.2 Envy-free and efficient equilibrium

As shown in the example of Figure 1, there are efficient equilibria that generate arbitrarily less revenue then any envy-free equilibrium. For the other direction, we show that all revenue-optimal efficient equilibria are envy-free.

Theorem 9 *For any AdAuctions instance such that $\alpha_i > \alpha_{i+1} \forall i$, all revenue-optimal efficient equilibria are envy-free. Moreover, we can write the revenue optimal efficient equilibrium explicitly as function of $\alpha, \mathbf{v}$ recursively as follows:*

$$b_n = \min\left\{v_n, \frac{\alpha_{n-1}-\alpha_n}{\alpha_{n-1}} v_{n-1}\right\},$$

$$b_i = \min\left\{v_i, \frac{\alpha_{i-1}-\alpha_i}{\alpha_{i-1}} v_{i-1} + \frac{\alpha_i}{\alpha_{i-1}} b_{i+1}\right\} \ \forall i < n.$$

PROOF. Given an efficient equilibrium $\mathbf{b}$, if it is not envy-free, we show that we can improve revenue by slightly increasing one of the bids. If the equilibrium is not envy-free, there is at least one player that envies the player above, i.e.

$$\alpha_i(v_i - b_{i+1}) < \alpha_{i-1}(v_i - b_i).$$

As pointed out in [7], if in an efficient equilibrium no player envies the above slot (i.e. no player i wants to take the above slot $i-1$ by the price per click player i is paying) then the equilibrium is envy-free.

Let i be the player with the smallest index that envies slot $i-1$. Consider the bid profile $\mathbf{b}'$ such that $b'_j = b_j$ for $j \neq i$ and $b'_i = b_i + \epsilon$. We will verify that the Nash inequalities for player $i-1$ still hold when $\epsilon > 0$ is sufficiently small. In other words, we will show that no Nash inequality for player $i-1$ holds with equality in $\mathbf{b}$.

For slots $j > i-1$, notice that

$$\alpha_j(v_i - b_{j+1}) \leq \alpha_i(v_i - b_{i+1}) < \alpha_{i-1}(v_i - b_i)$$

where the first is a standard Nash inequality and the second is the hypothesis that player i envies the above slot. Now, since $v_{i-1} > v_i$ in an efficient equilibrium, we have

$$\alpha_j(v_{i-1} - b_{j+1}) < \alpha_{i-1}(v_{i-1} - b_i).$$

For slots $j < i-1$, we use the fact that player i is the first envious player. Also, without loss of generality, we can assume player 1 bids v_1. Therefore we need to verify the Nash inequalities only for $j = 2, 3, \ldots, k-1$. We have

$$\alpha_{i-1}(v_i - b_i) \geq \alpha_j(v_i - b_{j+1}) > \alpha_j(v_i - b_j)$$

where the first inequality comes from the fact that player $i-1$ doesn't envy any player j above him and the second inequality comes from the fact that $b_j > b_{j+1}$, since otherwise the player in slot j would envy the player in slot $j-1$. This shows that the revenue optimal equilibrium is envy free.

To see that the bid profile defined in the theorem is optimal, we need to show the following things about this bid profile $\mathbf{b}$: (i) it is in Nash equilibrium, (ii) it is envy free, and (iii) no other efficient Nash equilibrium generates higher revenue. Begin by noticing that if $\mathbf{b}$ is Nash, then player $i-1$ doesn't want to take slot i, for all i, and therefore $\alpha_{i-1}(v_{i-1} - b_i) \geq \alpha_i(v_{i-1} - b_{i+1})$ and this is satisfied by definition by the bid vector presented. Notice also that this series of inequalities implies an upper bound on the maximum revenue in an efficient equilibrium and this bound is achieved exactly by the bid profile defined above.

Furthermore, for all $j \leq i-1$ we have $\alpha_{i-1}(v_j - b_i) \geq \alpha_i(v_j - b_{i+1})$ therefore by composing this expression with different values of i and j, it is straightforward to show that no player can profit by decreasing his bid. We prove that no player can profit by overbidding as a simple corollary of envy-freeness. For that, we need to prove that

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_{i-1}(v_i - b_i).$$

If $b_i = v_i$ than this is trivial. If not, then substitute the expression for b_i and notice it reduces to $v_{i-1} \geq v_i$. Now, this proved local envy-freeness, what implies that no player wants the slot above him by the price he player above him is paying. This in particular implies that no player wants to increase his bid to take a slot above. ■

5.3 Cost of efficiency: definition and example

Next we will analyze the relation between revenue and efficiency in GSP auctions.

We define the *cost of efficiency* for a given profile of click-through-rates as

$$\text{CoE}(\alpha) = \max_{\mathbf{v}} \frac{\max_{\mathbf{b} \in \text{Nash}(\alpha,\mathbf{v})} \mathcal{R}(\mathbf{b})}{\max_{\mathbf{b} \in \text{EffNash}(\alpha,\mathbf{v})} \mathcal{R}(\mathbf{b})}$$

where NASH is the set of all bid profiles in Nash equilibrium and EFFNASH is the set of all efficient Nash equilibrium.

First we give examples in which $\text{CoE}(\alpha) > 1$, in which case all revenue-optimal equilibria occur at inefficient equilibria. Our example will have $n = 3$ slots and advertisers. The click-through rates are given by $\alpha = [1, \frac{2}{3}, \frac{1}{6}]$ and the agent types are $v = [1, \frac{7}{8}, \frac{6}{8}]$. In this case, the best possible revenue generated by an efficient outcome is given by $\frac{1}{3} + \frac{7}{8} \approx 1.20833$ (this can be calculated using the formula in Theorem 9). However, for the (inefficient) allocation $\pi = [2, 1, 3]$, there is an equilibrium that generates revenue 1.21528.

In Figure 2 we calculate this value empirically for each $\alpha = [1, \alpha_2, \alpha_3]$, where each α_i is an integer multiple of 0.01 in $[0, 1]$. In all cases we found that $1 \leq \text{CoE}(\alpha) < 1.1$. The color of (α_1, α_2) in the graph corresponds to $\text{CoE}(1, \alpha_2, \alpha_3)$, where blue represents 1 and red represents 1.1. By solving a constrained non-linear optimization problem, one can show that the worst CoE for 3 slots is 1.09383.

5.4 Efficiency Versus Revenue when Click-Through-Rates are Convex

We now present a condition on α that implies $\text{CoE}(\alpha) = 1$. Our condition is that the click-through-rates are *convex*, meaning that $\alpha_i - \alpha_{i+1} \geq \alpha_{i+1} - \alpha_{i+2}$ for all i. We note that most models for CTRs studied in the literature satisfy convexity, such as exponential CTRs [15] and Markovian user models [1].

Theorem 10 *If click-through-rates α are convex then there is a revenue-maximizing Nash equilibrium that is also efficient.*

Our proof follows from a local improvement argument: given an instance with convex click-through-rates and an

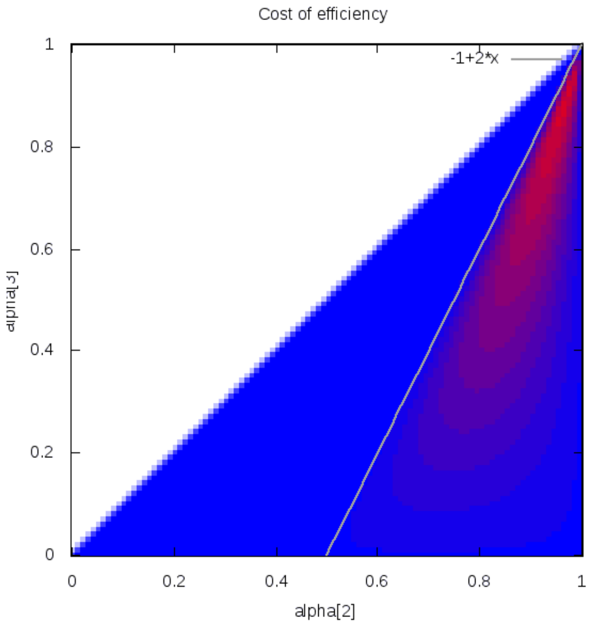

Figure 2: Cost of efficiency for $\alpha = [1, \alpha_2, \alpha_3]$: in the plot, blue means 1.0 and red means 1.1.

equilibrium that is not efficient, we show how to either improve it revenue or its welfare. A key step of the proof is bounding the maximum revenue possible in equilibrium for a given allocation, extending Theorem 9 to inefficient allocations.

PROOF. Let $\mathbf{b}$ be the revenue maximizing efficient Nash equilibrium. Fix an allocation π and let $\mathbf{b}'$ be an equilibrium under allocation π. We say that $\mathbf{b}$ is **saturated** for slot i if $b_i = v_i$. We start by presenting the proof of the theorem under the simplifying assumption that no slot is saturated in the maximum revenue equilibrium..

Under the no-saturation assumption, Theorem 9 implies

$$\mathcal{R}(\mathbf{b}) = \sum_i \alpha_i b_{i+1} = \sum_i \sum_{j \geq i} (\alpha_j - \alpha_{j+1}) v_j. \quad (2)$$

Notice that we can view this expression as a dot product of two vectors where one has elements of the form v_i and other has elements in the form $\alpha_j - \alpha_{j+1}$. Notice also that due to the convexity assumption, this is a dot product of two sorted vectors. Now, for $\mathbf{b}'$, we will bound revenue as follows. Define $m(\pi, i, j) = \max\{\pi(i), \pi(i+1), \pi(i+2), \ldots, \pi(j)\}$. Let p be such that the $k = i, i+1, \ldots, i+p$ are all the indices such that $m(\pi, i, k) = \pi(i)$. Now, notice that the player in slot i doesn't want to take slot $i+p+1$, so

$$\alpha_i (v_{\pi(i)} - b'_{\pi(i+1)}) \geq \alpha_{i+p+1} (v_{\pi(i)} - b'_{\pi(i+p+2)}).$$

This implies

$$\alpha_i b'_{\pi(i+1)} \leq \alpha_{i+p+1} b'_{\pi(i+p+2)} + (\alpha_i - \alpha_{i+p+1}) v_{\pi(i)}$$
$$= \alpha_{i+p+1} b'_{\pi(i+p+2)} + \sum_{j=i}^{i+p} (\alpha_j - \alpha_{j+1}) v_{m(\pi,i,j)}.$$

We can now apply recursion to conclude that $\alpha_i b'_{\pi(i+1)} \leq \sum_{j \geq i} (\alpha_j - \alpha_{j+1}) v_{m(\pi,i,j)}$, and hence

$$\mathcal{R}(\mathbf{b}') = \sum_i \alpha_i b'_{\pi(i+1)} \leq \sum_i \sum_{j \geq i} (\alpha_j - \alpha_{j+1}) v_{m(\pi,i,j)}. \quad (3)$$

Notice that equation (3) can also be written as a dot product between two vectors of type v_i and $\alpha_j - \alpha_{j+1}$. If we sort the vectors, we see that the $(\alpha_j - \alpha_{j+1})$-vector is the same in both (2) and (3). Moreover, the sorted vector of v_j for equation (3) is dominated by that of equation (2), in the sense that it is pointwise smaller. To see this, simply count how many times we have one of $v_1, \ldots, v_i$ appear in both vectors for each index i: for equation (2) they appear $\sum_{j=1}^{i} j$ times, whereas for equation (3) they appear at most $\sum_{j=1}^{i} 1 + \max\{p \mid m(\pi(j, j+p)) \leq i\} \leq \sum_{j=1}^{i} j$ times. Since the $(\alpha_j - \alpha_{j+1})$-vectors are the same in both equations, the v_i vector in the first equation dominates the order and in the first equation both vectors are sorted in the same order, so it must be the case that $\mathcal{R}(\mathbf{b}) \geq \mathcal{R}(\mathbf{b}')$.

It remains to remove our simplifying assumption about saturation and prove the general result. Let $\mathbf{b}$ be the optimal efficient equilibrium and let $S \subseteq [n+1]$ be the set of saturated bids, including $n+1$ (where we consider a "fake" player $n+1$ with $b_{n+1} = v_{n+1} = 0$), i.e., $i \in S$ iff $b_i = v_i$. Let $S(i) = \min\{j \in S; j > i\}$.

Given an allocation π, we wish to define an upper bound, $\overline{\mathcal{R}}_\pi$, on the revenue of a bid profile that induces allocation π at equilibrium. To this end, we define

$$B_\pi(j) = \begin{cases} \alpha_{S(j)-1} v_{S(j)} + \sum_{i=\sigma(j)}^{S(j)-2} (\alpha_i - \alpha_{i+1}) v_{m(\pi,\sigma(j),i)} \\ \quad \text{if } \sigma(j) \leq S(j) - 1 \\ \alpha_{S(j)-1} v_{S(j)} - v_j (\alpha_{S(j)-1} - \alpha_{\sigma(j)}) \\ \quad \text{if } \sigma(j) \geq S(j) - 1 \end{cases}$$

We then define

$$\overline{\mathcal{R}}_\pi = \sum_j B_\pi(j).$$

We claim that this is, indeed, an upper bound on revenue. Moreover, this bound is tight for revenue at efficient equilibria (i.e. when π is the identity id).

Claim 11 *If bid profile* $\mathbf{b}$ *induces allocation* π *at equilibrium, then* $\mathcal{R}(\mathbf{b}) \leq \overline{\mathcal{R}}_\pi$.

Claim 12 *There exists an efficient equilibrium with revenue* $\overline{\mathcal{R}}_{id}$.

Using these two claims we want to argue that id is the permutation that maximizes $\overline{\mathcal{R}}_\pi$ and therefore we can show that for all inefficient bid profile $\mathbf{b}'$ we have

$$\mathcal{R}(\mathbf{b}') \leq \overline{\mathcal{R}}_\pi \leq \overline{\mathcal{R}}_{id} = \mathcal{R}(\mathbf{b}).$$

To show this, consider some permutation π. Let $j = \max\{k : \pi(k) \neq k\}$ and define a permutation π' such that $\pi'(k) = k$ for $k \geq j$ and $\pi'(k) = \pi(k)$ for $k < \sigma(j)$ and $\pi'(k) = \pi(k+1)$ for $\sigma(j) \leq k < j$. Essentially this is picking the last player that is not allocated to his correct slot and bring him there. Now, if we prove that $\overline{\mathcal{R}}_{\pi'} \geq \overline{\mathcal{R}}_\pi$, then we are done, since we can repeat this procedure many times and get to id.

Claim 13 $\overline{\mathcal{R}}_{\pi'} \geq \overline{\mathcal{R}}_\pi$.

This completes the proof, subject to our claims, which we prove in Appendix C. ∎

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APPENDIX

A. BAYESIAN REVENUE WITH WELL-SEPARATED CLICK-THROUGH-RATES

Another way to bound the revenue of GSP in settings of incomplete information, without imposing reserve prices, is to assume that the slot click-through-rates are well separated, in the sense of [15]. We say that click-through-rates are δ-well separated if $\alpha_{i+1} \leq \delta\alpha_i$ for all i.

Lemma 14 *If click-through-rates are δ-well separated, then bidding $b_i(v_i) < (1-\delta)v_i$ is dominated by bidding $(1-\delta)v_i$.*

Proof. Suppose player i bids $b_i < (1-\delta)v_i$. If he increases his bid to $b_i' = (1-\delta)v_i$ then with some probability he still gets the same slot (event S) and with some probability he gets a better slot (event B). Then clearly $\mathbb{E}[u_i(b_i, b_{-i})|v_i] \leq \mathbb{E}[u_i(b_i', b_{-i})|v_i]$ since the expectation conditioned to S is the same and conditioned to B it can only increase by changing the bid to b_i'. To see that, let $\alpha_{\pi(i)}$ be the slot player i gets under b_i and $\alpha_{\pi'(i)}$ the slot he gets under b_i'. Conditioned on B we know that $\alpha_{\pi'(i)} \geq \delta^{-1}\alpha_{\pi(i)}$, and this generates value for bidder i of at least $\alpha_{\pi'(i)}(v_i - b_i')$, while the value with bid b_i was at most $\alpha_{\pi(i)}v_i$, which implies the claim:

$$
\begin{aligned}
E[u_i(b_i, b_{-i})|v_i, B] &\leq \mathbb{E}[\alpha_{\pi(i)}v_i|v_i, B] \leq \mathbb{E}[\delta\alpha_{\pi'(i)}v_i|v_i, B] = \\
&= \mathbb{E}[\alpha_{\pi'(i)}(v_i - (1-\delta)v_i)|v_i, B] \leq \\
&\leq \mathbb{E}[u_i(b_i', b_{-i})|v_i, B].
\end{aligned}
$$

■

Recall that under truthful bidding, the revenue of GSP is at least the revenue of VCG. If one eliminates the strategies

$b_i(v_i) < (1-\delta)v_i$ from the players strategy set, then it is easy to see that any Bayesian-Nash equilibrium $\mathbf{b}$ has high revenue.

Corollary 15 *If click-through-rates are δ-well separated, and all players play undominated strategies, then*

$$\mathbb{E}_{\mathbf{v}}[\mathcal{R}(\mathbf{b}(\mathbf{v}))] \geq (1-\delta)\mathbb{E}_{\mathbf{v}}[\mathcal{R}^{VCG}(\mathbf{v})].$$

Further, for any reserve price r, we also get

$$\mathbb{E}_{\mathbf{v}}[\mathcal{R}_r(\mathbf{b})] \geq (1-\delta)\mathbb{E}_{\mathbf{v}}[\mathcal{R}_r^{VCG}(\mathbf{v})].$$

Next we consider whether this bound on GSP revenue, with respect to the expected GSP revenue when all players report truthfully, continues to hold if agents do not eliminate dominated strategies. That is, we consider settings of limited rationality in which players may not be able to find dominated strategies. If we allow players to use dominated strategies, then we might have equilibria with very bad revenue compared to the expected revenue when agents bid truthfully, as one can see in the following example:

Example. Consider two players with iid valuations $v_i \sim \text{Uniform}([0,1])$ and two slots with $\alpha = [1, 1-\epsilon]$. Then VCG generates revenue $\mathbb{E}[\mathcal{R}^{VCG}(\mathbf{v})] = \mathbb{E}[\epsilon \min\{v_1, v_2\}] = O(\epsilon)$, and if agents report truthfully the GSP auction generates revenue $\mathbb{E}[\min\{v_1, v_2\}] = O(1)$. However, consider the following equilibrium:

$$b_1(v_1) = \begin{cases} \epsilon(1-\delta), & v_1 \geq \epsilon(1-\delta) \\ \epsilon v_1, & v_1 < \epsilon(1-\delta) \end{cases}$$

$$b_2(v_2) = \begin{cases} \epsilon, & v_2 \geq 1-\delta \\ \epsilon^2(1-\delta), & \epsilon(1-\delta) \leq v_2 < 1-\delta \\ \epsilon v_2, & v_2 < \epsilon(1-\delta) \end{cases}$$

It is not hard to check that this is an equilibrium. In fact, for two player GSP in the Bayesian setting, playing $(\alpha_1 - \alpha_2)v_i/\alpha_1$ is a best reply - and any bid that gives the player the same outcome is also a best reply. So, in the above example, one can simply check that the bids generate the same utility as bidding $b_i(v_i) = \epsilon v_i$. This example generates revenue $\mathbb{E}\mathcal{R}(\mathbf{b}) = O(\epsilon(\epsilon+\delta))$, so taking $\delta = O(\epsilon)$ in the above example give us $O(\epsilon^2)$ revenue. ■

However, this is a feature of having only 2 players, as shown in the following theorem, which is a version of Corollary 15 that doesn't depend on eliminating dominated strategies.

Theorem 16 *With n players with iid valuations v_i and δ-well separated click-through-rates, then for all Bayes-Nash equilibria $\mathbf{b}$ in which agents do not overbid,*

$$\mathbb{E}[\mathcal{R}(\mathbf{b})] \geq \frac{n-2}{n}(1-\delta)\mathbb{E}[\mathcal{R}^{VCG}(\mathbf{v})].$$

Proof. We will prove the stronger result that the expected GSP revenue at equilibrium is within a factor of $\frac{n-2}{n}(1-\delta)$ of the expected GSP revenue when agents report truthfully. We first claim that, for a profile $\mathbf{b}$ in Bayesian-Nash equilibrium and any two players i and j, we have that

$$\mathbb{P}_{v\sim F}[b_i(v) < (1-\delta)v - \epsilon, b_j(v) < (1-\delta)v - \epsilon] = 0.$$

To see this, suppose the contrary. Then there is $\epsilon' \ll \epsilon$ such that if we take $F' = F|_{[v^0-\epsilon', v^0+\epsilon']}$ then

$$\mathbb{P}_{v\sim F'}[b_i(v) < (1-\delta)v - \epsilon, b_j(v) < (1-\delta)v - \epsilon] > 0.$$

For ϵ' small enough $\underline{v_0} = v^0 - \epsilon$ and some $\epsilon'' < \epsilon$, we have

$$\mathbb{P}_{v\sim F'}[b_i(v) < (1-\delta)\underline{v_0} - \epsilon'', b_j(v) < (1-\delta)\underline{v_0} - \epsilon''] > 0.$$

Now pick v^i, v^j in this interval such that $\mathbb{P}_{v\sim F'}[b_i(v^i) \leq b_i(v) < (1-\delta)\underline{v_0}] > 0$ and the same for j. By lemma 14, playing $(1-\delta)v^i$ is a best response, then for player j for example, it can't be the case that any of the other players play between $b_j(v^j)$ and $(1-\delta)v^j$ with positive probability. Therefore

$$\mathbb{P}_{v\sim F'}[b_j(v) \in [b_i(v^i), (1-\alpha)v^i)] = 0$$

$$\mathbb{P}_{v\sim F'}[b_i(v) \in [b_j(v^j), (1-\alpha)v^j)] = 0$$

but notice this is a contradiction. This completes the proof of the claim.

Now, we can think of the procedure of sampling $\mathbf{v}$ iid from F in the following way: sample $v_i'' \sim F$ iid, let v_i' be the sorted valuations, and then apply a random permutation $\tau \in S_n$ to the values so that $v_i = v'_{\tau(i)}$. Notice that $\mathbf{v}$ is iid and now, notice that with $\geq 1 - \frac{2}{n}$ probability, v_i' and v'_{i+1} will generate $(1-\delta)v_i'$ and $(1-\delta)v'_{i+1}$ bids producing $(1-\delta)\alpha_i v'_{i+1}$ revenue, therefore

$$\mathbb{E}[\mathcal{R}(\mathbf{v})] \geq \mathbb{E}\left[\sum_i \left(1 - \frac{2}{n}\right)(1-\delta)\alpha_i v'_{i+1}\right]$$
$$\geq \frac{n-2}{n}(1-\delta)\mathbb{E}\left[\mathcal{R}^V(\mathbf{v})\right].$$

■

B. PROPHET INEQUALITY

For completness, we include a short proof of the Prophet Inequality, which was used in Lemma 3. The proof follows [11].

Theorem 17 (Prophet Inequality [11]) *Given independent random variables $z_1, \ldots, z_n$ if one defines t as the solution of the equation $t = \sum_{i=1}^n \mathbb{E}(z_i - t)^+$, then by defining $y_t = z_i$, where i is the smallest index such that $z_i \geq t$, and zero if $\max z_i < t$, then:*

$$\mathbb{E}[y_t] \geq \frac{1}{2}\mathbb{E}[\max_i z_i]$$

Proof. We can upper bound $\mathbb{E}[\max_i z_i]$ as :

$$\mathbb{E}[\max_i z_i] \leq t + \mathbb{E}[\max_i (z_i - t)^+] \leq$$
$$\leq t + \mathbb{E}[\sum_i (z_i - t)^+] = 2t$$

and lower bound $\mathbb{E}[y_t]$ as:

$$\mathbb{E}[y_t] = t\mathbb{P}(\max_i z_i \geq t) +$$
$$\sum_i \mathbb{E}[(z_i - t)^+ | \max_{j=1..i-1} z_j < t]\mathbb{P}(\max_{j=1..i-1} z_j < t) \geq$$
$$\geq t\mathbb{P}(\max_i z_i \geq t) + \sum_i \mathbb{E}[(z_i - t)^+]\mathbb{P}(\max_i z_i < t) = t$$

■

Notice that since t is increasing and $\sum_{i=1}^n \mathbb{E}(z_i - t)^+$ is decreasing, a solution always exists if each z_i has a distribution that has positive density everywhere. If this is not the case, the prophet inequality still holds by taking t to be either the supremum of $\{t : t \leq \sum_{i=1}^{n} \mathbb{E}(z_i - t)^+\}$ or the infimum of $\{t : t \geq \sum_{i=1}^{n} \mathbb{E}(z_i - t)^+\}$ (whichever results in larger $\mathbb{E}[y_t]$).

C. OMITTED PROOFS FROM SECTION 5

We now prove the claims from the proof of Theorem 10.

Proof of Claim 11 : We will show that for all $\mathbf{b}'$ inducing allocation π, we have $\alpha_{\sigma(j)} b'_{\sigma(j)+1} \leq B_\pi(j)$. For $\sigma(j) = S(j) - 1$, we use the fact that $b'_{\sigma(j)+1} = b'_{S(j)} \leq v_{S(j)}$. For $\sigma(j) < S(j) - 1$ the result follows in the same way as in the unsaturated case. For $\sigma(j) > S(j) - 1$, we use the fact that player j doesn't want to take slot j and therefore

$$\alpha_{\sigma(j)}(v_j - b'_{\sigma(j)+1}) \geq \alpha_{S(j)-1}(v_j - b'_{S(j)-1}) \geq \alpha_{S(j)-1}(v_j - v_{S(j)})$$

since

$$b'_{S(j)} \leq \min\{v_{\pi(1)}, \ldots, v_{\pi(S(j)-1)}\} \leq v_{S(j)}$$

and $\sigma(j) > S(j) - 1$ so one of the players with value $\leq v_{S(j)}$ must be among the first $S(j) - 1$ slots. Reordering the Nash inequalities above gives us the desired result. ■

Proof of Claim 12 : This claim follows from the formula defining the optimal-revenue efficient equilibrium in the previous section. ■

Proof of Claim 13 : Note first that $B_\pi(k) = B_{\pi'}(k)$ for all $k > j$. Moreover, for any k with $\sigma(k) < \sigma(j)$, we will have $\sigma'(k) = \sigma(k)$. In this case, either $S(k) < \sigma(k)$ in which case $B_{\pi'}(k) = B_\pi(k)$, or else

$$\begin{aligned} B_\pi(k) &= \alpha_{S(k)-1} v_{S(k)} + \sum_{i=\sigma(k)}^{S(k)-2} (\alpha_i - \alpha_{i+1}) v_{m(\pi,\sigma(k),i)} \geq \\ &\geq \alpha_{S(k)-1} v_{S(k)} + \sum_{i=\sigma'(k)}^{S(k)-2} (\alpha_i - \alpha_{i+1}) v_{m(\pi',\sigma'(k),i)} \\ &= B_{\pi'}(k). \end{aligned}$$

It remains to consider k is such that $\sigma(j) \leq \sigma(k) \leq j$; that is, those players k such that $\sigma(k) \neq \sigma'(k)$. For each such player, we will consider the difference between $B_\pi(k)$ and $B_{\pi'}(k)$. First note that, for player j, we have

$$\begin{aligned} & B_\pi(j) - B_{\pi'}(j) \\ &= \left(\alpha_{S(j)-1} v_{S(j)} + \sum_{i=\sigma(j)}^{S(j)-2} (\alpha_i - \alpha_{i+1}) v_{m(\pi,\sigma(j),i)} \right) \\ &\quad - \left(\alpha_{S(j)-1} v_{S(j)} + \sum_{i=\sigma'(j)}^{S(j)-2} (\alpha_i - \alpha_{i+1}) v_{m(\pi',\sigma'(j),i)} \right) \\ &= \sum_{i=\sigma(j)}^{j-1} (\alpha_i - \alpha_{i+1}) v_j \end{aligned}$$

For $k \neq j$, we claim that $B_{\pi'}(k) - B_\pi(k) \geq v_j(\alpha_{\sigma(k)-1} - \alpha_{\sigma(k)})$. We proceed by two cases. First, if $S(k) \leq \sigma(k)$, we have

$$\begin{aligned} B_{\pi'}(k) - B_\pi(k) &= \left(\alpha_{S(k)-1} v_{S(k)} - v_k(\alpha_{S(k)-1} - \alpha_{\sigma'(k)})\right) \\ &\quad - \left(\alpha_{S(k)-1} v_{S(k)} - v_k(\alpha_{S(k)-1} - \alpha_{\sigma(k)})\right) \\ &= v_k(\alpha_{\sigma(k)-1} - \alpha_{\sigma(k)}) \geq v_j(\alpha_{\sigma(k)-1} - \alpha_{\sigma(k)}) \end{aligned}$$

Second, if $S(k) - 1 > \sigma(k)$, then we have

$$\begin{aligned} & B_{\pi'}(k) - B_\pi(k) \\ &= \left(\alpha_{S(k)-1} v_{S(k)} + \sum_{i=\sigma'(k)}^{S(k)-2} (\alpha_i - \alpha_{i+1}) v_{m(\pi',\sigma'(k),i)} \right) \\ &\quad - \left(\alpha_{S(k)-1} v_{S(k)} + \sum_{i=\sigma(k)}^{S(k)-2} (\alpha_i - \alpha_{i+1}) v_{m(\pi,\sigma(k),i)} \right) \\ &= (\alpha_{S(k)-2} - \alpha_{S(k)-1}) v_{m(\pi',\sigma'(k),S(k)-2)} \\ &\quad + \sum_{i=\sigma'(k)}^{S(k)-3} v_{m(\pi',\sigma'(k),i)} [(\alpha_i - \alpha_{i+1}) - (\alpha_{i+1} - \alpha_{i+2})] \\ &\geq v_j(\alpha_{S(k)-2} - \alpha_{S(k)-1}) \\ &\quad + \sum_{i=\sigma'(k)}^{S(k)-3} v_j [(\alpha_i - \alpha_{i+1}) - (\alpha_{i+1} - \alpha_{i+2})] \\ &= v_j(\alpha_{\sigma(k)-1} - \alpha_{\sigma(k)}) \end{aligned}$$

Notice that we strongly use the fact that click-through-rates are convex in the last inequality to ensure that $(\alpha_i - \alpha_{i+1}) - (\alpha_{i+1} - \alpha_{i+2}) \geq 0$.

Therefore, taking the sum over all k with $\sigma(j) \leq \sigma(k) \leq j$, we have

$$\begin{aligned} \sum_{k:\sigma(j)<\sigma(k)\leq j} (B_{\pi'}(k) - B_\pi(k)) &\geq \sum_{i=\sigma(j)}^{j-1} v_j(\alpha_i - \alpha_{i+1}) \\ &= B_\pi(j) - B_{\pi'}(j) \end{aligned}$$

so that

$$\sum_{k:\sigma(j)\leq\sigma(k)\leq j} (B_{\pi'}(k) - B_\pi(k)) \geq 0.$$

Combining this with the fact that $B_{\pi'}(k) \geq B_\pi(k)$ for all k with $\sigma(k) < \sigma(j)$ or $\sigma(k) > j$, we conclude

$$\overline{\mathcal{R}}_{\pi'} = \sum_k B_{\pi'}(k) \geq \sum_k B_\pi(k) = \overline{\mathcal{R}}_\pi$$

as desired. ■